

OLD AND NEW THEORIES OF ECONOMIC GROWTH (I Part)

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Abstract: *In this article an attempt has been made to give comparative analysis of old and new theory of economic growth. The field of economic growth has become again very dynamic and very interesting after appearance of seminal Romer's 1986 and Lucas's 1988 articles, which initiated so called new theory of economic growth, sometime termed as theory of endogenous technological progress. This new theory, in some very important issues, stands in a sharp contrast with the old neoclassical version of theory of economic growth, which similarly can be termed as the theory of exogenous technological progress. Apart from the introduction and the concluding section, core of the article is presented in four sections. In first of them exposition of old version of neoclassical growth theory is given. In the following 3 sections survey of new theory is given. Version that eliminates assumption of diminishing returns to capital is discussed first. Then, version that uses human capital as engine of growth is presented. After that, models that use R&D as engine of growth is discussed. Models with spillovers from international trade are also shortly presented.*

Abstract: *U ovom članku je učinjen pokušaj da se da komparativna analiza stare i nove teorije privrednog rasta. Disciplina ekonomskog rasta je ponovo postala vrlo dinamična i interesantna nakon pojavljivanja odlučujućih članaka Romera iz 1986 i Lukasa iz 1988, koji su inicirali takozvanu novu teoriju privrednog rasta, ponekad nazivanu i teorija endogenog tehnološkog progresu. Ova nova teorija, u nekim vrlo važnim pitanjima, stoji u oštrom kontrastu sa starom neoklasičnom verzijom teorije privrednog rasta, koja slično može biti nazvana teorija egzogenog tehnološkog progresu. Pored uvodnog i zaključnog odeljka jezgro članka je dato u četiri odeljka. U prvom od njih izložena je stara verzija neoklasične teorije privrednog rasta. U naredna tri odeljka dat je pregled nove teorije rasta. Verzija koja eliminiše opadajuće prinose na kapital je prva prezentirana. Zatim, je data verzija kod koje ljudski kapital predstavlja mašinu rasta. Nakon toga je data verzija kod koje su ulaganja u istraživanje i razvoj mašina rasta. Modeli sa prelivanjima iz međunarodne trgovine su takođe ovde kratko prezentirani.*

Key words: *Economic growth, Endogenous models, Diminishing returns, Human capital*

Ključne reči: *ekonomski rast, endogeni modeli, opadajući prinosi, ljudski kapital*

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In this article we will attempt a comparative analysis of old and new theory of economic growth and an assessment of recent developments in the field of the theory of economic growth. Direct motivation for this undertaking comes from the fact that the field of economic growth has become again very dynamic and very interesting in last fifteen years or so. More precisely, this new interest has been revived by Romer's 1986 and Lucas's 1988 articles, which initiated so called new theory of economic growth, sometime termed as theory of endogenous technological progress. This

new theory, in some very important issues, stands in a sharp contrast with old neoclassical version of theory of economic growth, which similarly can be termed as theory of exogenous technological progress. In fact, very initiation and development of new theory can, in a large extend, be seen as an effort to find solutions for some, more empirical than theoretical, problems, which couldn't have been solved by old theory. It is, therefore, impossible to give an account of new theory of growth without first giving appropriate survey of old theory, and this practice is followed in this article.

Two things should be kept in mind while reading this article. *First* of all, article is more survey of ideas and concepts related to old and new theory of growth, than detailed and meticulous survey of all relevant literature in the field. It is, in other word, much more story about general ideas that distinguish new from old theory of growth than detailed survey of all contributions in the field. Literature in this field, especially in the so cold new theory of growth, is so enormous that it would not be possible to discuss it in short paper like this one. Consequently, just most important, cornerstone articles are explicitly mentioned and discussed. *Second*, in order to make those concepts and ideas easier to understand and especially easier to compare among themselves and with other concepts and ideas, numerous simplifications and modifications of their original expositions are performed. Reader should be ware of this fact, and especially of the fact that some of those modifications, although appropriate in the context in which they are used in the article, would be inappropriate in some other context and for some other purposes.

Apart from this introduction and concluding section, core of the article is given in following four sections. In first of them exposition of old version of neoclassical growth theory is given. In the following sections survey of new theory is given. Version that eliminates assumption of diminishing returns to capita is discussed first. Than, in following chapter version that uses human capital as engine of growth is presented. After that models that use R&D as engine of growth is discussed. Models with spillovers from international trade are here also shortly presented.

1. Old neoclassical theory of economic growth

I. Although first works of modern style growth theory can be traced back to the early forties contribution of Jan Tinbergen (1942) and later efforts of Fabricant (1954), Abramovitz (1956), Kendrick (1956) and some other authors, it is Robert Solow's article from 1956th that is regarded as corner stone of neoclassical growth theory. Theoretical model developed at that article and it's later econometric application [Solow, R. (1957)] represented significant departure from, at that time prevailing, Harod-Domar model of economic growth. It also represented significant advancement compared to the old model of growth: it is based on more realistic assumptions (constant vs. zero elasticity of factor substitution), it is more manageable, and, consequently, it has greater explanatory power than Harod-Domar model. In addition, owing to its simplicity and to the fact that it makes possible measuring of different factors contribution to the economic growth, model immediately became very popular and for a long time very influential in economic profession.

Model is based on the concept of an *aggregate production function*. Aggregate output of the economy (gross domestic product) is, according to that concept, determined by the quantity of conventional factors of production, that is quantity of labor and capital, on the one side, and by the efficiency of prevailing "technology" in respecting economy, on the other side. Latter concept, efficiency of "technology", is more commonly termed as global (or total, or combined) factor productivity, or simply, and probably most appropriately, as a residual. First important characteristic of the model and underlying production function, and it's important shortcoming, is that it takes technological efficiency to be given *exogenously*: it does not explain evolution of technology. Second characteristic of production function is common assumption of *constant return to scale*: simultaneous proportional increase of all conventional factors is followed by equal increase of output. Third is characteristic of *decreasing marginal productivity of both conventional factors of production* (labor and capital): proportional increase of, for example, capital is, *ceteris paribus* (that is holding labor and

technology constant), followed by less than proportional increase of output, and similarly for labor increase. Diminishing marginal productivity (return) to capital is especially important: it, first, limits the ability of Solow's model of growth to give satisfactory explanation of cross-countries differences in income per capita level, and, second, it limits its ability to give full explanation of differences in rate of growth either. Those are key features that distinguish the traditional view of economic growth from the new generation of growth models, which form a body of knowledge of endogenous theory of economic growth.

Above described characteristics of technology are most commonly expressed in the form of Cobb-Douglas production function which is written as $Y_t = A_t K_t^a L_t^{(1-a)}$ $1 > a > 0$

or more commonly in its' labor-augmenting form as $Y_t = K_t^a (A_t L_t)^{(1-a)}$ $1 > a > 0$ (1)

where Y_t stands for aggregate output, K_t for aggregate capital, L_t for labor, and A_t for technological efficiency. Technological efficiency is assumed to grow at some predetermined rate and it is most commonly expressed as exponential function of rate of technological progress (g_A) and time (t), or more formally $A_t = \exp(g t)$. It is often preferable to use per capita transformation of Cobb-Douglas production function, which has a form

$$q_t = \frac{Y_t}{A_t L_t} = K_t^a (A_t L_t)^{-a} = \left[\frac{K_t}{A_t L_t} \right]^a = l_t^a \quad (2)$$

where, obviously, q_t stands for per capita output expressed in efficiency units ($Y_t / A_t L_t$), while l_t presents corresponding capital labor ratio ($K_t / A_t L_t$). For the sake of simplicity in this case we assume that labor force and population are the same.

Coefficients a and $(1-a)$ express *elasticity of production* with respect to capital and labor respectively. Constant return to scale is represented by the fact that sum of those two coefficients is equal to 1. It is very often assumed in empirical investigation that, owing to perfect competition on the factors market, marginal productivity of labor [$Y_{L_t} = dY / dL = (1-a)K^a (AL)^{-a} A$] and capital [$Y_{K_t} = dY / dK = aK^{(a-1)} (AL)^{(1-a)}$] should be equal to their price that is to wages (w), in a case of labor, and profits (r), in a case of capital. If this is the case, than a and $(1-a)$ also represent share of labor and capital in gross domestic product, or more formally

$$\frac{r_t K_t}{Y_t} = \frac{Y_{K_t} K_t}{Y_t} = \frac{a K^{(a-1)} (AL)^{(1-a)} K}{K^a (AL)^{(1-a)}} = a \quad \text{and} \quad \frac{w_t L_t}{Y_t} = \frac{Y_{L_t} L_t}{Y_t} = \frac{(1-a) K^a (AL)^{-a} AL}{K^a (AL)^{(1-a)}} = (1-a) \quad (3)$$

It is intuitively clear that those two coefficients can be also used as a measures of diminishing marginal return to each input: the smaller is a value (that is, the larger is $(1-a)$ value), the smaller will be return to capital investment, and similarly for labor, the smaller is $(1-a)$ (that is, the larger is a value), the smaller will be reward to increasing employment.

Important parameter for description of technology, and especially for measurement of speed of diminishing return, is *elasticity of substitution* (e). It presents ratio of relative changes in factors proportions (K / L) over relative changes of marginal rate of substitution (F_L / F_K), or that often happen to be the same, relative changes of relative factors prices (w / r) that cause that change in factors proportions. Formally

$$e = \frac{\frac{\Delta(K/L)}{(K/L)}}{\frac{\Delta(Y_L/Y_K)}{(Y_L/Y_K)}} = \frac{\frac{\Delta(K/L)}{(K/L)}}{\frac{\Delta(w/r)}{(w/r)}} \quad (4)$$

In the case of Cobb-Douglas production function elasticity of substitution is equal to one. It follows directly from the constancy of a and $(1-a)$. In fact, constancy of a and $(1-a)$ implies constancy of their ratio

$$\frac{1-a}{a} = \frac{Y_t L_t / Y_t}{Y_t K_t / Y_t} = \frac{Y_t L_t}{Y_t K_t} = \frac{w_t L_t}{r_t K_t} \quad (5)$$

and it can be true only if relative changes in relative factors prices (w/r) are followed by the relative changes of factors proportions (K/L) of a same magnitude and opposite direction. Obviously, this is a case of unit value of elasticity of substitution. More interestingly, if we assume elasticity of substitution to be less than one (and it is, at least at a short run, much more realistic case), than we can expect effect of diminishing returns to develop much faster than in a case of Cobb-Douglas production function. In limiting case of Harod-Domar production function we have zero elasticity of substitution, and consequently instantaneous effect of diminishing returns. On the other hand, if elasticity of substitution is larger than one, diminishing return will develop slower. In limiting case of unlimited elasticity (Abramovicz, 1956), which is absolutely unrealistic, effect of diminishing return is supposed to vanish. Obviously, in both of these cases ($e > 1$ and $e < 1$) above expressed relation of factors shares will not be constant, like in a case of Cobb-Douglas function, but will change depending on the assumption on elasticity of substitution.

The idea of diminishing return to capital is most commonly shortly expressed by two assumptions related to marginal product of capital and by so-called Inada conditions. Assumptions regarding marginal product of capital are formally expressed as

$$Y_{K_t} = \frac{dY}{dK} = \frac{a(AL)^{(1-a)}}{K^{(1-a)}} > 0 \quad \text{and} \quad \frac{d^2 Y}{dK^2} = \frac{-(1-a)a(AL)^{(1-a)}}{K^{(2-a)}} < 0 \quad (6)$$

First assumption says that marginal product of capital should always be positive, while second assumption says that its' second derivative should be negative meaning that it decline with increase of capital. Inada conditions on the other hand imply

$$\lim_{K \rightarrow +\infty} \frac{dY}{dK} = 0 \quad \text{and} \quad \lim_{K \rightarrow 0} \frac{dY}{dK} = \infty \quad (7)$$

First expression says that marginal product of capital tend to zero if capital tend to infinity, while second expression says that it tend to infinity as capital tend to zero.

One way to describe behavior of this kind of economy is to use so called *behaviouristic approach*. Here we assume constant rate of investment ($I_t/Y_t = s_t = s$) and constant rate of depreciation (δ) and look how capital and GDP develop over time. Capital stock represents a sum of gross investments committed in the past that are still in use in present moment (t). Crucial assumption regarding development of capital stock here is assumption of constant rate of investment. So, capital stock accumulate over time throughout net investment in a way that can be presented formally as

$$\dot{K}_t = sY_t - \delta K_t \quad (8)$$

Assuming CD production function given in (1), above expression can be developed to

$$\dot{K}_t = sK_t^a (A_t L_t)^{(1-a)} - \delta K_t \quad (9)$$

so that rate of growth of capital becomes

$$g_K = \frac{\dot{K}}{K} = sK_t^{a-1} (A_t L_t)^{(1-a)} - \delta = s \left[\frac{K_t}{A_t L_t} \right]^{(a-1)} - \delta = s l_t^{(a-1)} - \delta \quad (10)$$

Since $(a-1) < 0$, it follow that first part of expression (10) decrease until it reach point at which

$$s l_t^{(a-1)} = \delta \quad (11)$$

From previous relations and assuming that L_t and A_t do not change, $g_L = 0$ and $g_A = 0$, it also follows that at steady state

$$\begin{aligned}
 sK^a (AL)^{(1-a)} &= \delta K \\
 K^* &= AL \left[\frac{s}{\delta} \right]^{\frac{1}{1-a}} \\
 g_K &= \frac{\dot{K}}{K} = 0
 \end{aligned}
 \tag{12}$$

where K^* presents steady state level of capital.

Substituting in expression (1), and again assuming that $g_L=0$ and $g_A=0$, we get following relations for steady state of GDP (Y^*)

$$\begin{aligned}
 Y^* &= \left[AL \left(\frac{s}{\delta} \right)^{\frac{1}{1-a}} \right]^a (AL)^{(1-a)} = AL \left[\frac{s}{\delta} \right]^{\frac{a}{1-a}} \\
 g_Y &= \frac{\dot{Y}}{Y} = 0
 \end{aligned}
 \tag{13}$$

Graphic presentation of this economy behavior is given at Figure 1. Diminishing returns to capital imply that output grows slower than capital. In other words, production function is concave in the (Y,K) space. Consequently saving function sY is also concave meaning that saving / investment grows slower than capital depreciation. So, economy eventually reaches steady state where depreciation equals saving, while capital, and in that way output grows at zero rate of growth.

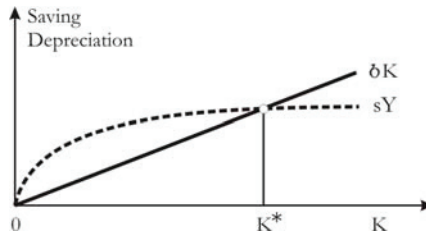


Figure 1.

II. Another more comprehensive approach in analyzing this kind of economy is so called *optimizing version model approach*. Here it is assumed that immortal representative consumer maximizes present discounted value of the utilities of his present and future consumption streams. Rate of saving is here in that way not taken as given but is rather determined by the model of consumer behavior. His preferences over consumption streams are described by:

$$\text{Max} \int_0^{\infty} U(C_t) e^{-\rho t} dt \quad \rho > 0 \tag{14}$$

where C_t presents real consumption stream, $U(C_t)$ is utility function, while ρ stands for appropriate discount rate. Utility function most commonly used in growth analysis is of the form

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} \quad \sigma > 0 \tag{15}$$

where σ present coefficient of risk aversion. This is known as constant-relative-risk-aversion (CRRA) utility function. Coefficient of relative risk aversion is given by:

$$-\frac{C \frac{d^2 U(C)}{dC^2}}{\frac{dU(C)}{dC}} = \sigma$$

It is reciprocal to elasticity of intertemporal substitution. When σ tend to zero utility function become almost linear in C_t while when it approach one utility function tend to $\ln C_t$. If $\sigma < 0$ than $C^{(1-\sigma)}$ is increasing in C_t . On the other hand if $\sigma > 0$ than $C^{(1-\sigma)}$ is decreasing in C_t . So,

dividing $C^{(1-\sigma)}$ with $(1-\sigma)$ ensure that marginal utility of consumption $\frac{dU}{dC} = C^{-\sigma}$ is positive regardless of the magnitude of the value of σ .

In describing budget constraint we will, for the sake of simplicity, assume zero rate of depreciation ($\delta=0$) and will have following budget constraint

$$\begin{aligned} Y_t &= C_t + \dot{K}_t \\ \dot{K}_t &= Y_t - C_t = K_t^a (A_t L_t)^{(1-a)} - C_t \end{aligned} \quad (16)$$

Resource allocation problem is now solved by maximizing utility function (14) subject to the budget constraint (16). This is equivalent to maximizing current value Hamiltonian of the form

$$H_t = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta_t [K_t^a (A_t L_t)^{(1-a)} - C_t] \quad (17)$$

An optimal allocation must maximize H at each point of time t , provided that current-value of capital accumulation, θ_t , is chosen correctly. The solution is obtained under following first order and co-state conditions.

$$\text{First-Order Condition is } \frac{dH_t}{dC_t} = 0 \quad (18)$$

$$\begin{aligned} \frac{dH_t}{dC_t} &= \frac{(1-\sigma)C_t^{-\sigma}}{1-\sigma} - \theta_t = 0 & C^{-\sigma} &= \theta \\ \frac{\dot{C}}{C} &= -\frac{1}{\sigma} \frac{\dot{\theta}}{\theta} \end{aligned}$$

It says that at every moment the shadow price of investment θ_t , must be equal to the marginal utility of consumption. It means that at every moment of time output will be best allocated between investment and consumption if the marginal gain from the unit increase of consumption is equal to marginal loss of unit decrease in investment.

$$\text{Co-State Condition is given by } \frac{dH_t}{dK_t} = \rho\theta_t - \dot{\theta} \quad (19)$$

$$\begin{aligned} a\theta K^{(a-1)}(AL)^{(1-a)} &= \rho\theta_t - \dot{\theta} \\ \frac{\dot{\theta}}{\theta} &= \rho - aK^{(a-1)}(AL)^{(1-a)} \end{aligned}$$

This is the Fisher equation that must hold at every moment of time. It says that the sum of the marginal product of capital and the capital gain per unit of capital must be equal to the pure rate of time preference.

$$\text{Finally, transversality Condition is } \lim_{t \rightarrow \infty} e^{-\rho t} \theta_t K_t = 0 \quad (20)$$

It says that the value of capital must tend to zero. It determines which of one-parameter family of solutions given by (18) and (19) for one initial conditions $K(0)=K_0$ is the right one.

We are looking for the values of K_t , θ_t , and C_t such that their growth rates are constant. What we get is known as balanced growth path. Solution is given in equations (16), (18) and (19).

$$\text{Expressions (18) and (19) give together } g_C = \frac{1}{\sigma} [aK^{(a-1)}(AL)^{(1-a)} - \rho] \quad (21)$$

Since ρ and σ are constant, equation (21) imply that balanced growth path with positive growth rate requires marginal product of capital ($aK^{(a-1)}(AL)^{(1-a)}$) to be constant and above the value of ρ . Formally, it means that $\dot{g}_C = 0$ must hold, and it itself imply that

$$d[aK^{(a-1)}(AL)^{(1-a)}] / dt = 0 \quad \text{and that} \quad (a-1)g_K + (1-a)(g_A + g_L) = 0 \quad (22)$$

We now can imagine two situations: one without and another with technological progress.

(A) Solow's Model Without Technological Progress.

We will first assume zero rate of technological progress – $g_A=0$. For the sake of simplicity we will also assume zero rate of labor force – $g_L=0$. In this case equation (22) become $g_K = \frac{\dot{K}}{K} = 0$ (23)

Growth rate of capital at balanced growth path is, obviously, in this particular case equal to zero. Now, having in mind above assumptions and results, and log-differentiating CD production function (1) we get following result $g_Y = \frac{\dot{Y}}{Y} = ag_K + (1-a)(g_A + g_L) = 0$ (24)

Rate of growth of aggregate output is also at balanced growth path equal to zero.

Having in mind that $\frac{Y}{K} = \frac{Y_K}{a} = \frac{1}{a} \frac{dY}{dK} = \frac{C + \dot{K}}{K}$

and using result from expression (21) we get following results $(\sigma g_C + \rho) / a = K^{(a-1)} (AL)^{(1-a)}$

$$(\sigma g_C + \rho) / a = K^a (AL)^{(1-a)} / K = \frac{Y}{K} \tag{25}$$

Now, knowing from above results that $g_Y=0$ and $g_K=0$ we conclude from (25) that $(\sigma g_C + \rho) / a$ must be constant, and that therefore growth rate of consumption must be zero – $g_C=0$.

Since we also assumed that labor force (L_t) is constant ($g_L=0$), it is obvious that in this case per capita variable must grow at same growth rate as their aggregate counterparts. Therefore, balanced growth path solution of Solow's model without technological progress and with constant population is given by $g_C = g_Y = g_K = g_c = g_y = g_k = 0$ (26)

where, as we know $y = Y / L$, $c = C / L$, and $k = K / L$.

Obviously this economy exhibits long-run zero rate of growth because of diminishing return to capital. As we know, marginal productivity of capital is given by $Y_K = \frac{dY}{dK} = \frac{a(AL)^{(1-a)}}{K^{(1-a)}}$ and it decreases as capital increase: under our assumptions numerator is constant, while denominator increase.

(B) Solow's Model With Technological Progress.

Now we will assume positive rate of technological progress – $g_A>0$. Again, for the sake of simplicity we will also assume zero growth rate of labor force – $g_L=0$. In this particular case equation (22) gives following result $g_K = g_A$ (27)

As before, from (1) we get $g_Y = ag_K + (1-a)(g_A + g_L) = g_A$ (28)

Following these results and having in mind equation (25) we conclude that consumption must also grow at the same rate. So, we get $g_Y = g_C = g_K = g_A$ (29)

Again, since we assumed zero rate of growth of labor force, it follows that per capita variables must grow at the same rate as their aggregate counterparts. Formally

$$g_Y = g_K = g_C = g_A = g_y = g_k = g_c \tag{30}$$

Another important property of this economy is that at balanced growth path, its' rate of saving and investing should be constant. Using results from (25) and having other above given considerations it can be proved in following way

$$s = \frac{I_t}{Y_t} = \frac{\dot{K}_t}{Y_t} = \frac{(\dot{K} / K)}{(Y / K)} = \frac{g_K}{(Y_K / a)} = \frac{ag_K}{(dY / dK)} = \frac{ag_K}{(\sigma g_K + \rho)} \tag{31}$$

Since a , g_K , σ , and ρ are all constant it follows that saving / investment rate must also be constant.

Shortly, when $g_A > 0$, output per capita grows in the long run at positive constant rate. Owing to positive technological progress, capital can grow in the long run without decreasing its marginal productivity. In other words technological progress compensates diminishing returns to capital. The marginal productivity of capital can therefore be permanently sustained above ρ allowing in that way positive per-capita growth. Looking now again at the expression for marginal productivity of capital, $Y_K = \frac{dY}{dK} = \frac{aA^{(1-a)}L^{(1-a)}}{K^{(1-a)}}$, we can see that for L constant, $g_L=0$, and $g_K=g_A$, marginal productivity of capital is held constant. This is what delivers positive per-capita growth in the long run in the neoclassical growth model developed by Solow. Obviously, exogenously given technological progress is source of sustained per-capita growth in the long run in this model of growth. Since it is given exogenously it is not explained by the model. It is why this kind of model is often called exogenous growth model.

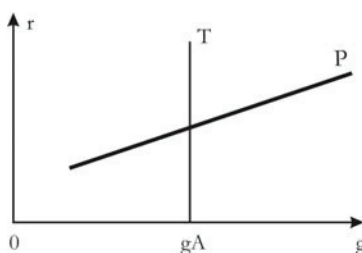


Figure 2.

In figure 2 we represented above Solow's model in the (r, g) space. Here, since we assumed zero rate of depreciation, interest rate, r , is equal to the marginal productivity of capital. Curve P represent preferences that is the equation (21) which here describe the consumption side of economy in terms of balanced growth paths expressed in pairs of (r, g) . On the other hand curve T presents technology, that is production side of economy in terms of balanced growth paths given by pairs (r, g) . As can be seen, equilibrium growth rate, intersection of two curves, is given by the exogenously given variable g_A .

III. An economy described in above way has some striking growth characteristic. Assuming for the moment, for the sake of simplicity, that population / labor (L) grows at constant rate of growth ($g_L > 0$) and that there is no growth in the level of technology (A constant, $g_A = 0$), **first** important feature can be stated as follow: *for a given saving / investment rate, and therefore, for a given rate of accumulation of physical capital, per capita output (Y/L) and per capita amount of capital (K/L) will reach a steady state or constant value.* Simplest possible explanation is that as per capita capital stock grows, the return to capital falls (because of the effect of diminishing return), so that the amount of new investment per capita increase but at diminishing rate (because of slower growth of output, whose constant fraction is supposed to be equal to new investment, that is caused again by diminishing return to capital). Eventually the amount of new investment per capita will drop to the level equal to the rate of capital deterioration and then growth of per capita capital stocks will become equal to zero. Consequently output per capita will also grow at zero growth rate. On the other hand aggregate output and capital will grow at the rate of growth of population / labor force (g_L). If we now relax assumption referring to the level of technology and allow rate of technological progress to be positive (A grow, $g_A > 0$), than, following the above considerations, we conclude that at a *long run* per capita value of capital and output grow at the rate that is equal to the rate of technological progress, while value of aggregate output and aggregate capital grow at the rate that is equal to the sum of the rate of technological progress and the rate of growth of population.

Fact that aggregate output and aggregate capital grows at the same rate also implies that *capital output ratio grows at zero rate of growth: capital coefficient is, in other word, constant.* But,

since we know that capital share in income is constant, than *capital coefficient constancy implies constancy of profit rate, that is zero rate of growth of profit rate*. Next, since income increased due to *technological progress should be distributed between profit and wages, zero rate of growth of profit rate implies equality between rate of growth of technological progress and rate of growth of wages*. Finally, this fact, together with constancy of labor share in income, *implies equality between rates of growth of wages, labor productivity and technological progress*. It is pretty obvious that in, the above given, case of Cobb-Douglas production function those relations hold not only for the long run, steady state, but also for the short run state. It is less obvious, but it can be proven, that in the steady state those relations hold for all other forms of production functions [Mankiw (1995)]. It is also important to note that all above predictions of the model are more or less supported by long-range statistical records for developed countries.

Second important implication of traditional model of growth refers to long run effect of investment / saving rate: *although, following previous paragraph, we can conclude that saving / investment rate is of no importance for long run growth rate of per capita output and capital, it is of a crucial importance for the level of per capita output and capital, and, therefore, for the level of standard of living in some economy*. Or, to put it in a more formal manner: *saving / investment rate crucially determine level of growth paths, but it have no influence on the slope of (natural) logarithm of growth path (growth rate), whatsoever*. So, countries with higher investment / saving rate are, *ceteris paribus*, supposed to have higher per capita output and, consequently higher standard of living in the steady state of growth. Explanation is simple and straightforward: higher investment or saving rate result in more capital accumulated per worker, which according to previous expressions increase per capita output of the economy. Therefore, this model suggests that *sustained, long run differences in the level of per capita income across nations can in a large degree be explained by differences in saving rates*. Note, however, that, since higher saving rate do not mean higher long run rate of growth, *differences in the long run growth rate of output per capita among nations cannot be explained by differences in saving rate*. The *influence of population rate of growth on the level of income per capita is opposite*: larger increase of population leads, following our analysis, to lower level of capital per capita and, in that way, to lower level of long run income per capita.

Third characteristics refer to the *growth path during the transition from one to another steady state*. Assume that for some reason, let say because of change in property rights structure, society become more saving oriented and that its saving ratio increase. According to previous considerations, new higher rate of saving will lead in the long run to higher capital labor ratio, and in this way will enable workers to produce more output per worker and to increase standard of living. In order to get this higher standard of living, economy must for certain period of time grow faster than its' long run growth rate. However, once the new steady state is reached, per capita income will continue to grow at its long run rate of growth, which is equal to rate of growth of technological progress. In fact, increase in saving ratio shifts economy from the lower level long run growth path to the higher level long run growth path. This movement, however, is not instantaneous: certain period of adjustment, known as *transitory period*, is necessary for this to happen. Slope of both long run growth paths (i.e. of their natural logarithm), lower (before transition) and higher (after transition), have same slope, meaning that their rates of growth are equal. Therefore, transitory growth path, which transit economy from the lower to the higher path, should have higher slope, meaning higher rate of growth. In the case of decreasing saving rate situation is opposite: in order to reach lower growth path from higher one economy should grow at rate which is smaller than long run growth rate. In both case, however, *speed of adjustment*, and it is important property of a Solow's model, *is determined by the parameter a , in other words it crucially depend on the degree of diminishing returns*.

We have seen earlier that, according to neoclassical model, differences in saving / investment rate explain, together with other factors (set of technological possibilities, population growth, capital depreciation rate), differences in the level of output per capita and standard of living across countries.

We can add now, that according to Solow's theory of growth, *differences in the rate of growth across countries are result of their different position on transitory path*, caused by changes in saving rate or, more precisely, by possible changes in factors that determine saving / investment rate.

Forth important property of Solow's model is connected with previous one, and is known as *convergence property: countries that have access to similar production technologies (production possibilities), exhibit same degree of capital depreciation, have same rate of growth of population and have same saving / investment rate should converge to the same steady state levels of per capita income*. More specifically: poor country (with lower level of capital / labor ratio), which increase saving rate to the level same as at some target rich country, will grow faster during the transition period as it catches up to the rich country; however, if all enumerated conditions are satisfied, both countries will ultimately arrive at the same standard of living. All those conditions, not only equality of saving rate, should be satisfied in order to get convergence to the same stationary level of income. If any one of those other conditions (same technology frontiers, same rate of capital depreciation, same rate of growth of population) is not fulfilled, than countries with same rate of saving will converge indeed, but to its own specific steady state growth path of income per capita, which will differ from other countries. It is quite natural and expected because level of development, as we know, is not determined solely by saving rate. This is known as *conditional convergence*. [Mankiw (1995)].

All this is, again, direct result of effect of diminishing return, and applies both in the case of closed and in the case of open economy. However, this *convergence is much faster in the case of open economy*. Since poor countries have lower capital / labor ratio, the return to investment in physical capital will be higher than in rich countries (again because of diminishing returns). So, poor country is supposed to be more attractive for capital from abroad, and this capital will enter in poor country, either in the form of direct investment, portfolio investment or in the form of borrowing. Capital stocks and capital / labor ratio are likely to grow more quickly in this case than in the case of closed economy, thereby speeding up the process of convergence.

IV. All those 4 properties of neoclassical model and neoclassical theory of growth in general are not problematic so much from the theoretical point of view. More problematic is empirical question: *can the model give appropriate explanation for the wide variation in development experience recorded in the world*. As a matter of fact, Solow's neoclassical model of growth has come under attack recently just because of not being able to provide *an empirically adequate theory of growth*. It is, therefore, worth shading little bit more light on the empirical part of neoclassical growth story.

Decomposition of the economic growth into the portion due to expansion of conventional factors (labor and capital) and portion due to technological progress has been, and still is one of the most attractive and most exiting outcomes of Solow's growth theory. **First**, important characteristic of this growth accounting practice is *low level of elasticity of output with respect to capital, coefficient a , and consequent strong effect of diminishing return on capital*: value of a coefficient usually used in different empirical research was about 1/3. Basically this value has been calculated on two different ways. First is econometrical, where regression equation techniques have been used to estimate production functions parameters. Second is based on above expression (3), where factors shares in gross domestic product as given in national accounting systems have been used as estimation of a . Interestingly enough, value of approximately same magnitude (1/3) has been almost uniformly obtained independently of estimation technique used. More interestingly, approximately same share of capital in gross domestic product of 30% has been found in vast variety of countries. This fact by itself seeks appropriate explanation, and as we know such an explanation has not been provided yet.

Second, this kind of measurements show that increase in capital labor ratio can explain just 10-20% of long run growth rate of per capita output. Remaining 80-90% are left unexplained and are simply termed as technological progress, global (or total, or combined) factor productivity growth, or simply, and probably most appropriately, as *residual*. Unfortunately, Solow's theory has nothing to say neither about anatomy of this residual, nor about policy that might influence it. It is especially

important when we know magnitude of this residual. No doubt, something of so large magnitude (80-90% ?!) cannot be just taken as exogenously given: it requires detailed and profound explanation. This shortcoming is especially striking in the face of fact that, in this way measured, global factor productivities vary over time very much. Those variations, obviously, also need appropriate explanation. Common characteristic, which is particularly interesting, of those variations for almost all developed countries is acceleration of growth in the period 1950-1974 compared to previous 50 years, and deceleration of growth after 1974.

Third important property and problem of those measurements refers to the fact that, above mentioned, *property of sharply diminishing return on capital places sharp limit on models ability to explain cross-country differences in per capita income*. If we assume that $a=1/3$, than doubling of capital stock per capita will increase steady state of income by just 26% $[(2^{1/3}-1)\times 100 = 26]$. Obviously, large differences in capital per capita produce small differences in output per capita. Or to put it in world reality context: in order to explain by capital accumulation fact that output per capita in USA is 20 times larger than that in Kenya, the capital stock per capita in the United States would have to be 8 000 times larger than that in Kenya. And we know, from Summers and Heston (1991), that USA capital labor ratio is only 26 times that of Kenya. Model faces similar difficulties even when we compare USA with more similar countries than Kenya.¹ It is quite obvious from above expressions, but it can be proven more rigorously², that this property and behavior of the model crucially rest on the magnitude of a : with larger value for a more cross-countries variation in the level of income per capita can be explained by variation in capital stock per capita.

Forth property and shortcoming of those measurements is that, owing again to the sharp diminishing returns to capital, *model is unable to explain cross-countries differences in the rates of growth by just referring to transitory dynamics and country's position on transitory path*. Model predicts that if, for example, some country decide to increase its saving rate by 50%, than country's per capita income in new steady state will increase by about 22%. Since country's transition period is, according to Plosser (1992), somewhere around 30 years, it means that on average annual growth rate during transition period will be larger than long run growth rate just for 0.7% per year. Obviously, large increase in saving rate produces small increase in (transitory) growth rate.

Fifth problem is in fact other side of this same coin: of course, it is problem of the *length of transition period* (or length of adjustment or convergence period). Model predicts, according to Mankiw (1995), that annual rate of adjustment should be about 4% per year. Statistical data indicate, however, that the rate of adjustment is as large as half of that value, and that convergence period might be about twice of that implied by estimated rate of adjustment. Again, models estimate of the speed of adjustment and length of transitory period crucially rest on the assumption on the magnitude of a . Increase of a by twice, from $1/3$ to $2/3$, would bring estimated rate of convergence and length of transition period to the magnitudes that are in accordance with real statistical data. And upward correction of value of elasticity of production with respect to capital, that is increase of a , is exactly what has been proposed recently by some authors.³

Final, **sixth** empirical difficulty of Solow's model refers to its' *inability to explain cross-countries differences in rate of returns on capital*. This model predicts that, in the case of Cobb-Douglas production function, which assume unit elasticity of substitution and $a=1/3$, typical poor country, which has ten times smaller income per capita than typical rich country, should have about one hundred times larger rate of return to capital than that in rich countries. More specifically, since

¹ Above example is taken from Plosser (1992).

² For rigorous proof and elaborate consideration of this problem in the case of general production function see Mankiw (1995).

³ For more detailed discussion on those problems, and especially on the question of speed of adjustment, see Mankiw (1995), Mankiw et al (1992), Plosser (1992), Baro (1992).

average rate of return in rich countries is about 10%, model predict that rate of return in poor countries should be about 1000%. It more than anything else contradicts to what we have in reality.⁴

V. As we have already stressed, all above mentioned theoretical and empirical problems of neoclassical growth theory originate from two main sources of difficulties. **First** source refers to the fact that capital exhibits extremely strong decreasing marginal productivity, or what is most of the time same thing, strong diminishing returns. In this way it runs in the trouble of not being able to explain cross-country differences not only in the level of per capita income and standard of living, but also in the growth rates. **Second** source of difficulties is in exogenous nature of the rate of technological progress and steady-state growth. The theory does not provide any explanation of technological progress anatomy, and does not give any clue for a possible role of economic policy in this field. Long-term growth is independent of saving ratio and is determined simply by exogenously given rate of technological progress.

It is, therefore, not suppressing that main efforts of so-called new theory of growth have been concentrated on the above two sources of difficulties. Avoiding and overcoming those difficulties would help overcoming other problems steaming from them. So, efforts have been made to brake the constrained of diminishing return to capital accumulation, on the one side, and to internalize in the model of growth very process of technological progress, or in other words to make the technological progress endogenous, on the other side. All those efforts can be, broadly speaking, divided in 3 groups.

First group of the efforts have been oriented toward direct and simple elimination, from the production function, of the assumption of diminishing returns to capital. Sometimes this is done by, first, augmenting concept of capital to include other forms of capital like human capital, and than by eliminating diminishing returns on this broadly conceived capital (Rebelo, 1991). Sometimes, however more radical approach that eliminate diminishing returns solely on physical capital is applied (Jones and Manuelli, 1990). Obvious problem with this approach is that it assumes exactly what it wants to arrive at.

Second group of the efforts was built under assumption that human capital accumulation present engine of economic growth. First and still most prominent model of this sort is one proposed by Lucas (1988). Human capital accumulation is what guarantee positive long run growth rate and what compensates for diminishing returns on physical capital.

Finally, **third** and in our opinion most sophisticated groups of the efforts are those that are often related to the notion of R&D-based or idea-based growth models (Romer, 1990; Grosman and Helpman, 1991; Aghion and Howitt, 1992). Needless to say human capital also play crucial role in this kind of models. However, like in Sollow model it is technological progress that is here regarded as engine of growth. But, whereas in Sollow model technological progress is given exogenously in this group of models it is given endogenously. It is result of deliberate R&D activity in society.

Most of those different approaches have been very often followed with two kinds of relaxing assumptions that additionally enriched our understanding of the growth process and especially of our understanding of different policy consequences for the growth process. First one refers to the redefinition of very concept of capital, while second one refers to introduction of concept of externality and spillovers in growth analysis. *Redefining concept of capital* is done in way that understanding and measurement of reproducible capital has been broadened to include all different form of capital. First of all, it was broadened to include so-called *human capital* that is to include all those investment that somehow, by improving the quality of labor, are being embodied in labor force. They include: improvements that come from investment in formal and informal education and

⁴ Again result is extremely dependent on value of a . But it is also very dependent on the value of elasticity of substitution. Reasonable increase of a from $1/3$ to $2/3$ and increase of elasticity of substitution from 1 to 4, which can be appropriate for large and / or very open countries, can bring predictions of the model to the realistic level. For more details see Mankiw (1992).

training, from improvement in health of population, from immigrants acculturation (adjustment) and other forms of attitude changes, and so on. Second, concept of *R&D capital* was also added. It refers to improvement in the quality of new products and productive processes as a result of investment in R&D activity and all sorts of innovations and inventions involved with it. There are some other form of capital that can broaden concept of capital even more, but those two are no doubt fare more important than other, and have been consequently most extensively explored.

Note however that broadening concept of capital cannot by itself bring sustained positive long run rate of growth. What brings it is assumption that at least some forms of capital goods can be produced entirely with reproducible inputs, that is without use of nonreproducible inputs. By allowing that at least some subset of capital can be produced without nonreproducible inputs, models make possible increase of steady state growth rate as a result of increase in the rate of investment in respecting subset of capital. Investments again, like in Harod-Domar pre-Solow's era, are important determinant of steady-state growth rate. Consequently, long run rate of growth is again very sensitive to all those policy measures that have influence on rate of investment in different form of capital (tax structure, structure of government expenditure, property right transparency and transferability, political stability, democracy and so on).

Second, some authors have been concentrated on the fact that *there are spillovers, externalities or public goods features to some forms of capital*. Owing to this property, investors in those forms of capital (most commonly R&D capital and human capital) are not able to capture all effects and benefits produced by their investment. Some benefits are appropriated by other actors of the game. In other words, there are discrepancies between private and social / total rate of returns on those investments: social rates are larger than private rates of return. For that reason, level of investment in those forms of capital is beyond its optimal level if investment decisions are being made entirely by private agents and without governmental involvement. Consequently, and for purpose of growth analysis more importantly, while private returns may be diminishing, total social returns may not exhibit that property due to spillovers and externalities.

It is, however, again very important to stress that it is not spillovers and external effects that makes possible sustained growth in those models. What makes possible sustained growth in those models is a fact that at least some subset of capital or some forms of capital can be produced without nonreproducible inputs. In other words, what makes possible sustained growth is property of constant return to scale in all inputs that can be accumulated, and this property is brought here by the spillovers. In this respect, obviously, this group of models does not differ from previous one. But, by adding externalities and spillovers, they make long run growth rate even more sensitive with respects to economic policy. More importantly, by giving more elaborate and detailed picture of growth, they are able to give more sophisticated recommendations of policy measures for increase of long run growth rate.

In what follow, we will give an analytical review of above-mentioned three groups of endogenous growth models.

2. Endogenous models with nondecreasing returns to capital

I. First model of this sort to be discussed here is so called "AK" model used first time by Rebelo (1991). Nondecreasing returns on capital are in this model introduced simply by, first, augmenting concept of capital to include all forms of nonconventional capital, most notably human capital and R&D capital, and second, by assuming unit elasticity of production with respect to capital defined in that way.

Concept of *human capital* is by itself very broad: it covers all sorts of improvement in labor efficiency, no meter whether they came from increase in labor education and skill, from improvement in health of labor force and population in general, from acculturation of emigrants, and so on. Of course, all those improvement in quality and efficiency of labor forces are conditional to

corresponding type of investment. Consequently, *human capital should be conceived and measured* as cumulative of those investment committed in the past, which still have influence on (marginal) productivity of labor. Investment in formal and informal education and training, together with sacrifices (opportunity cost of workers and employers) that make possible learning by doing, are with no doubt most important and predominant part of investment in human capital. For that reason, in most of the literature, especially in those empirically oriented, very notion of human capital refer to capital developed in the education and training process.

Of course concepts of human capital and investment in education are old one. They were, for the first time formally and explicitly, introduced almost five decades ago in the works of Schultz, Backer, Hansen, Mincer, Blaug, and others. Approximately at the same time, education and human capital were introduced in the economic growth theory. Those early as well as later efforts [Denison E. (1867), (1985), Schultz T. (1962), Pasharopoulos G. (1985), Kendrick J. (1973), (1981), Jorgenson (1995), Jorgenson and Griliches (1967)] were mainly concerned with contribution of education and human capital to economic growth. They present part of so called *sources of growth analysis*, whose main aim is decomposition and explanation of Solow's residual. Contribution of education to economic growth is expressed in all those studies in, more or a less, the same way: total contribution of labor force is decomposed into part that measure contribution of "raw" labor (unskilled part of all works) and total contribution of education (skilled part of all works); latter, total contribution of education, is expressed as a sum of contributions of all levels and types of education; contribution of certain level or type of education (high school, university education and so on) is, further, expressed as multiplication of rate of growth of that type of education and its' elasticity of production; measurement of elasticity of production with respect to corresponding level of education is, on the other hand, based on assumption that marginal productivity of that level of education is proportional to its' rental, that is to differences between wages of two successive levels of education. Contributions of the other parts of human capital can be measured in a similar manner. More formally, this result can be obtained using a sort of production function similar to one given in previous expressions, except that labor input is measured, not by number of employee or their hours of works, but by quality or efficiency adjusted labor index. This index is almost always calculated as a weighted sum of quantity of different kind of labor (education), where relative level of wages and salaries of those kinds of labors (education) are used as a wages. Above described approach in analysis of contribution of human capital is often called labor-augmenting approach. Recently, some authors [Mankiw. N. G. (1995), Mankiw, N. G., Romer, D., and Weil D. N. (1992)] proposed another way of implementing human capital in analysis of growth, which accordingly can be termed as capital augmenting approach. Concept of capital is here augmented to include human capital while elasticity of production with respect to capital is increased to include not only share of physical but also share of human capital in GDP. Needless to say, both, labor-augmenting and capital-augmenting approaches belong to the tradition of neoclassical growth theory. They both still keep assumption of decreasing returns to capital although concept of capital is defined in different way than in original Solow's model.

II. Models with *endogenous growth* differ from both above discussed version of traditional approach in its ability to *brake the link between diminishing returns and capital accumulation*, and in this way to make long run growth rate dependent on saving / investment rate. As already mentioned, the simplest and most often quoted way to do it is one proposed by Rebelo (1991). Basically, in this model capital is treated in a same way as in above mentioned capital-augmenting approach: it is augmented in a way to include, apart from conventional physical capital, all forms of human capital and all forms of intangible capitals. Production function has a property of constant return to scale, just like traditional one, but it no longer show diminishing return to capital accumulation. It can be formally expressed like

$$Y_t = A_t K_t \tag{32}$$

where K now, presents broadly defined capital (physical, human and other intangible capital). In per capita terms it has form $y_t = A_t k_t$ (33)

where k_t now stand for newly defined capital labor ratio, $k_t = K_t/L_t$.

We can regard this kind of production function as a special case of one given in the previous consideration where $(1-a)=0$. In other words it can be regarded as a special case of *Cobb-Douglas* production function. However, and it is not emphasized in literature, it can also be regarded as a special case of *Horod-Domar* production function

$$\begin{aligned} Y_t &= \min[fK_t, qL_t] \\ q_t &= Y_t / L_t \\ f_t &= Y_t / K_t = A_t \end{aligned} \quad (34)$$

where assumption of persistent capital shortage is accepted, so that we always have that $Y_t = f_t K_t = A_t K_t$ as in expression (32). Of course, again, capital is here defined in a way to include human capital and other forms of intangibles.

Most important property of this model is in a fact that its long run rate of growth, as well as short run, depend on saving / investment rate. It can be most easily seen from expression for capital per capita rate of growth which now has following form

$$\begin{aligned} \dot{K}_t &= sA_t K_t - \delta K_t \\ g_k &= \frac{\dot{K}_t}{K_t} = sA_t - \delta \end{aligned} \quad (35)$$

As we can see, owing to nondiminishing return to capital, saving line $sA_t K_t$, never cross depreciation line δK_t . Obviously if $sA_t > \delta$ we will have positive long run growth of aggregate capital and in that way for aggregate output. If we assume unchanged labor force, $g_L = 0$, capital per capita rate of growth will behave same way. It will make long run capital per capita rate of growth dependent on saving / investment rate. Capital per capita rate of growth is defined with rate of saving / investment, as well as with rate of disinvestments (deterioration) and rate of technological progress.

Substituting in expression (33) shows that long run output per capita rate of growth should grow at a same rate. So, long run output per capita and standard of living rate of growth can be increased by increase of saving / investment rate. Consequently, anything that can influence people to save and invest more in any kind of capital can also help an increase in the long run rate of growth. Tax policy is obvious example. Transparency and transferability of property rights is another. Development and regulation of financial market next. But, so are mandatory elementary education, free access to secondary and postsecondary education, free medical care and other measures that stimulate investment in human capital. And so on.

III. Above described model of growth assume that total capital, (physical, human, R&D, and all forms of intangibles) does not show diminishing return. Following two models are based on more radical assumption that physical capital itself does not show diminishing returns. Sustained positive economic growth is here based on dropping standard neoclassical assumption of diminishing return to physical capital. One interesting way of doing it is done by Jones and Manuelli (1990). They developed production function that violate first Inada conditions and in that way bring sustained endogenous growth. Somewhat modified (Ribeiro, Maria-Joao, 2003) their production function can be presented as $Y_t = vK_t + K_t^a (A_t L_t)^{(1-a)}$ (36)

Obviously standard conditions related to marginal productivity of capital,

$$\frac{dY}{dK} = v + \frac{a(AL)^{(1-a)}}{K^{(1-a)}} > 0 \quad \text{and} \quad \frac{d^2Y}{dK^2} = \frac{-(1-a)a(AL)^{(1-a)}}{K^{(2-a)}} < 0, \quad \text{are here preserved.}$$

Same applies for second Inada condition

$$\lim_{K \rightarrow 0} \frac{dY}{dK} = \infty$$

However, first Inada condition is violated in this production function as can be seen below

$$\lim_{K \rightarrow +\infty} \frac{dY}{dK} = v > 0 \quad (37)$$

Capital develops as usual and can be described using following expression

$$\dot{K} = sY - \delta K = svK + sK^a (AL)^{(1-a)} - \delta K \quad (38)$$

where s presents rate of saving / investment, while δ presents rate of depreciation.

The behavior of economy can further be described with following expression for the rate

$$\text{of capital growth } g_K = \frac{\dot{K}}{K} = \frac{sY}{K} - \delta = s \frac{vK + K^a (AL)^{(1-a)}}{K} - \delta = sv + s \frac{(AL)^{(1-a)}}{K^{1-a}} - \delta \quad (39)$$

If we now assume that labor force and technology do not change, that is $g_L=0$ and $g_A=0$, than, obviously, as K goes to infinity, middle part of this expression will tend to zero, capital rate of growth will be equal to rate of growth of capital per capita, and will tend to $g_K = g_k = sv - \delta$

In this situation positive sustained per-capita growth is possible if $sv > \delta$.

This can be shown graphically (see Figure 3.). Having in mind expression (37), saving function sY will tend asymptotically to the line svK . If this line is above δK line, and it will be above it if $sv > \delta$, than depreciation will never reach the level of saving, and economy can sustain positive capital and output rate of growth forever. Under assumption of unchanged labor force and technology ($g_L=0$ and $g_A=0$) per-capita capital and output rates of growth will be equal to its aggregate counterparts.

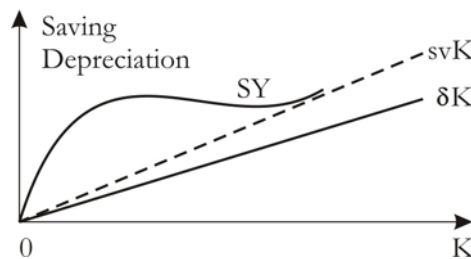


Figure 3.

IV. Above is given bibehavioristic version of this model. *Optimizing version*, which do not applies constant saving rate but instead derive it from optimizing behavior of consumers, can be developed in similar manner as before for Sollow's model of growth. Resulting per-capita growth

$$\text{rate is given by } g_y = g_k = g_c = \frac{1}{\sigma} \left(\frac{dY}{dK} - \delta - \rho \right) \quad (40)$$

In this circumstances, since $\lim (dY/dK)=v$ (when K goes to infinity) sustained positive per capita growth is possible if $v > \delta + \rho$.

5. Baro and Sala-i-Martin (1995 page 172) also developed model that eliminate diminishing return to capital. Model is based on constant return to scale production function and the one-sector-model in which output can be used for consumption, investment in physical capital, and investment in human capital. Somewhat modified (Ribeiro, Maria-Joao, 2003) production function has following form $Y_t = K_t^a (A_t H_t)^{(1-a)}$ (41)

where H_t present human capital engaged in production. It can be presented as $H_t = L_t h_t$

where L_t stand for labor input while h_t present human capital per worker. If we assume unchanged L_t and A_t , that is $g_A=0$ and $g_L=0$, than total human capital will grow only as the result of growth of average human capital per capita.

The budget constraint in this economy is of the following form $Y_t = C_t + I_{Kt} + I_{Ht}$ (42) where I_{Kt} presents investment in physical capital and I_{Ht} stands for investment in human capital. Two types of capital will develop according to the following two equations

$$\begin{aligned} \dot{K}_t &= I_{Kt} - \delta K_t \\ \dot{H}_t &= I_{Ht} - \delta H_t \end{aligned} \quad (43)$$

Assuming same utility function as in the previous case and constraints given in (42) and (43), *current value Hamiltonian* for representative agent is now given by

$$J_t = \frac{C_t^{1-\sigma}}{1-\sigma} + u_t [I_{Kt} - \delta K_t] + v_t [I_{Ht} - \delta H_t] + w_t [K_t^a (A_t H_t)^{(1-a)} - C_t - I_{Kt} - I_{Ht}] \quad (44)$$

where u_t and v_t present current value of physical capital and human capital accumulation respectively, while w_t stands for the Lagrangian multiplier referring to budget constraint.

Solution is given by the following sets of first order and co-state conditions. First order conditions are given below. For consumption we will have following solutions

$$\frac{dJ_t}{dC_t} = 0 \quad (45)$$

$$\frac{\dot{C}}{C} = -\frac{1}{\sigma} \frac{\dot{w}}{w}$$

For investment in physical capital we get following relations $\frac{dJ_t}{dI_{Kt}} = 0$ (46)

$$u = w$$

Finally, for investment in human capital we get

$$\frac{dJ_t}{dI_{Ht}} = 0 \quad (47)$$

$$v = w$$

Co-state equations are, for physical capital

$$\frac{dJ_t}{dK_t} = \rho u_t - \dot{u}_t \quad , \quad (48)$$

$$\frac{\dot{u}}{u} = \rho + \delta - a K_t^{(a-1)} (A_t H_t)^{(1-a)}$$

and for human capital

$$\frac{dJ_t}{dH_t} = \rho v_t - \dot{v}_t \quad (49)$$

$$\frac{\dot{v}}{v} = \rho + \delta - (1-a) K_t^a (A_t H_t)^{-a}$$

Since from above $u=v=w$, we get following requirements $\frac{\dot{u}}{u} = \frac{\dot{v}}{v}$, which, having in mind above relations, implies

$$a K^{(a-1)} A^{(1-a)} H^{(1-a)} = (1-a) A K^a A^{-a} H^{-a} \quad (50)$$

$$\frac{K}{H} = \frac{a}{1-a}$$

By Euler equation⁵ growth rate, which is also equal to per-capita growth rate (because of $g_L=0$), is given by $g = \frac{1}{\sigma} [a K^{(a-1)} (A H)^{(1-a)} - \delta - \rho]$ (51)

$$= \frac{1}{\sigma} [A^{(1-a)} a^a (1-a)^{(1-a)} - \delta - \rho]$$

⁵ Derivation of Euler equation is given in Appendix at the end of the paper.

Now, using these relationships, especially relationship (50), output can be presented in the following way $Y = K^a (AH)^{(1-a)} = K^{(a-1)} (AH)^{(1-a)} K$

$$Y = A^{(1-a)} \left(\frac{1-a}{a} \right)^{(1-a)} K = BK \quad (52)$$

Obviously, if we assume $g_A=0$ than B must be constant. This implies that $\frac{d^2Y}{dK^2} = 0$ and it is violation of second condition related to marginal productivity of physical capital. Consequently this model also generates sustained positive long run growth. This is here done via elimination of the assumption of the diminishing returns to physical capital, which is brought by constant (K_t/H_t) ratio. Note also that like AK model this model do not have transitional dynamics. Its long run and short run rate of growth is always equal to expression (51).

3. Endogenous models with human capital

I. In Lucas (1988) model of growth human capital is engine of growth. It specifies human capital accumulation, which allow for endogenous growth. Lucas assumes that there are L_t workers in economy whose skill level, h , ranges from zero to infinity. For the sake of simplicity we will assume that L_t does not change, that is $g_L=0$. In that way human capital accumulation can be highlighted more clearly. Worker with h level of skill and one unit of time devotes the fraction of $u(h)$ of his active time to production and $[1-u(h)]$ fraction of active time for his own human capital accumulation.

To present Lucas model in the way comparable to previous models we will start with following simplification of his original production function⁶ (Ribeira, Maria-Joao, 2003)

$$Y_t = K_t^a (A_t L_t^e)^{(1-a)} \quad (53)$$

where L_t^e present effective workforce. It can formally be given with

$$L_t^e = \int_0^{\infty} u(h)L(h)h dh \quad (54)$$

Note that if all workers are identical with skill level h and time allocation u , than $L_t^e = uhL_t$, and production function become $Y_t = K_t^a (A_t uhL_t)^{(1-a)}$

$$\text{Physical capital development can be expressed in usual way } \dot{K}_t = Y_t - C_t \quad (55)$$

On the other hand, human capital accumulates according to production function for skill producing sector of the following form $\dot{h}_t = h_t b(1-u_t)$ (56)

where b presents efficiency parameter of this production function. As can be seen this sector of production is very human capital intensive: there is no physical capital in it.

Note that for human capital development Lucas use linear function proposed first by Uzawa (1965). It means that this production do not exhibits diminishing return. This specification is what allows for sustained per-capita growth at constant rate. We can already see from (56) that at balanced path rate of growth of aggregate and per-capita human capital should be

$$g_h = \frac{\dot{h}_t}{h_t} = b(1-u_t) \quad (57)$$

and that u_t should be constant ($u_t=u$).

⁶ Original specification of production function was $Y_t = A_t K_t^a L_t^{e(1-a)} h_{at}^\gamma$, where the term h_{at}^γ captures the externalities related to human capital. This last term is no doubt important for analysis of policy implications. However it is not crucial for explanation of long run growth properties of Lucas model.

Now problem of optimization can be presented as one in which consumer maximize his present value of utility function, given in previously given expressions (14) and (15), subject to constraints given in expressions (55) and (56). Current value Hamiltonian for the solution of this

$$\text{problem is } H_t = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta_{1t} [K_t^a (A_t u_t h_t L_t)^{(1-a)} - C_t] + \theta_{2t} [h_t b_t (1 - u_t)]$$

First order conditions are given below. For C_t first order condition provide following solution

$$\begin{aligned} \frac{dH}{dC} &= 0 \\ C^{-\sigma} &= \theta_1 \\ \frac{\dot{C}}{C} &= -\frac{1}{\sigma} \frac{\dot{\theta}}{\theta} \end{aligned} \quad (58)$$

For allocation of time we get following solutions on the basis of first order condition

$$\frac{dH_t}{du_t} = 0 \quad (59)$$

$$\theta_1 (1-a) A h L K^a (A u h L)^{-a} = \theta_2 b h$$

Co-state condition for physical capital ensure solution

$$\begin{aligned} \frac{dH_t}{dK_t} &= \rho \theta_{1t} - \dot{\theta}_{1t} \\ \frac{\dot{\theta}_1}{\theta_1} &= \rho - a K^{(a-1)} (A u h L)^{(1-a)} \end{aligned} \quad (60)$$

while for human capital we get

$$\frac{dH_t}{dh_t} = \rho \theta_{2t} - \dot{\theta}_{2t} \quad (61)$$

$$\theta_1 (1-a) A u L K^a (A u h L)^{-a} + \theta_2 b (1-u) = \rho \theta_2 - \dot{\theta}_2$$

In balanced growth path solutions must be such that C , K and h grow at constant rate.

The prices θ_1 and θ_2 decline at constant rate and the time allocation variable u is constant.

Equations (58) and (60) together give

$$g_C = -\frac{1}{\sigma} \frac{\dot{\theta}_1}{\theta_1} = \frac{1}{\sigma} \{ a K^{(a-1)} (A u h L)^{(1-a)} - \rho \} \quad (62)$$

where first part in bracket present marginal productivity of capital. Obviously, as in solution for Sollow model, constant g_C imply constant marginal productivity of physical capital.

With constant A , L (by assumption) and u (by model) constancy of marginal productivity of capital imply

$$\begin{aligned} \frac{\dot{Y}_K}{Y_K} &= \left(\frac{d\dot{Y}}{dK} \right) / \left(\frac{dY}{dK} \right) = 0 \\ (a-1)g_K + (1-a)g_h &= 0 \\ g_K &= g_h \end{aligned} \quad (63)$$

Having this in mind and using production function (53 and 53') and its rate of growth we get

$$\begin{aligned} Y_t &= K_t^a (A_t u_t h_t L_t)^{(1-a)} \\ g_Y &= a g_K + (1-a) g_h = g_h \end{aligned} \quad (64)$$

Finally, having in mind assumptions of constant A and L , we conclude that in this model should be $g_C = g_Y = g_K = g_c = g_y = g_k = g_h$ (65)

In this model, obviously, sustained per-capita growth is obtained not via technological progress, as in Sollow's model, but via sustained human capital accumulation. In other words, in

Lucas model physical capital can be accumulated without decreasing its' marginal productivity because human capital is also growing at the same rate as physical capital. It can be seen more clearly if we take closer look at expression for marginal productivity of capital

$$Y_K = \frac{dY}{dK} = aK^{(a-1)}(AuhL)^{(1-a)} = \frac{a(AuL)^{(1-a)}h^{(1-a)}}{K^{(1-a)}}$$

Looking at last transformation of this expression we see that, since by assumption $g_L=0$, $g_A=0$ and $g_u=0$, it should be constant because of equality of rates of growth of physical and human capital.

If this is so, than it is of utmost importance to identify how human capital growth is determined within model. We should determine engine of growth, g_h . We already have expression (56) for human capital development. By log-differentiating expression (59) we get

$$\begin{aligned} \frac{\dot{\theta}_1}{\theta_1} + \frac{\dot{h}}{h} + a \frac{\dot{K}}{K} - a \frac{\dot{h}}{h} &= \frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{h}}{h} \\ \frac{\dot{\theta}_1}{\theta_1} &= \frac{\dot{\theta}_2}{\theta_2} \end{aligned} \quad (66)$$

Now comparing equations (59) and (61) we can establish following relationship

$$\theta_1(1-a)AhLK^a(AuhL)^{-a} = \theta_1(1-a)AuLK^a(AuhL)^{-a}x\frac{u}{h}$$

Now combining two equation we get

$$\begin{aligned} \frac{\theta_2 b h u}{h} + \theta_2 b - u \theta_2 b &= \rho \theta_2 b - \dot{\theta}_2 \\ \frac{\dot{\theta}_2}{\theta_2} &= \rho - b \end{aligned} \quad (67)$$

Finally, on the basis of (66) and (67) we are getting rate of human capital accumulation

$$g_h = g_k = g_c = g = -\frac{1}{\sigma} \frac{\dot{\theta}_1}{\theta_1} (b - \rho) \quad (68)$$

So, we can conclude that, first, in this model of endogenous growth, growth rate increases with effectiveness of investment in human capital (b). Second, economic growth is negatively related to the preferences parameters ρ and σ .

APPENDIX

1. EULER EQUATION

Suppose that *representative individual maximizes* present (discounted) value of his utility function, that is

$$\text{Max} \int_0^{\infty} U(C_t) e^{-\rho t} dt ; \quad U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} \quad \sigma > 0$$

subject to the restriction $\dot{S}_t = rS_t + w_t - C_t$ where S_t stands for assets in general, w_t is wage rate, r is interest rate, while other symbols have same meaning as in the main body of the text.

The *current value Hamiltonian* is $H_t = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta_t [rS_t + w_t - C_t]$

The first order condition gives

$$\frac{dH_t}{dC_t} = \frac{(1-\sigma)C_t^{-\sigma}}{1-\sigma} - \theta_t = 0$$

$$C^{-\sigma} = \theta$$

$$g_C = \frac{\dot{C}}{C} = -\frac{1}{\sigma} \frac{\dot{\theta}}{\theta} = \frac{1}{\sigma} g_\theta$$

On the other hand co-state condition gives

$$\frac{dH_t}{dS_t} = \rho\theta_t - \dot{\theta}$$

$$r\theta_t = \rho\theta_t - \dot{\theta}$$

$$g_\theta = \frac{\dot{\theta}}{\theta} = \rho - r$$

Together first-order and co-state conditions give familiar Euler equation $g_C = \frac{1}{\sigma} g_\theta = \frac{1}{\sigma} (\rho - r)$ which presents intertemporal preference structure for representative consumer adopted in this article.

2. SOCIAL PLANNER SOLUTION IN ROMER'S MODEL

In order to judge about welfare properties of Romer's (1990) we are supposed to consider social planner formulation of this model. The social planner maximize the representative consumer's utility

$$\text{Max} \int_0^{\infty} U(C_t) e^{-\rho t} dt ; \quad U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} \quad \sigma > 0 \quad (1)$$

c, L_A

subject to the following constraints

$$Y_t = K_t^a L_{Y_t}^{(1-a)} A_t^{(1-a)} \quad (2)$$

$$\dot{K}_t = Y_t - C_t \quad (3)$$

$$\dot{A}_t = bL_{A_t} A_t \quad (4)$$

$$L_t = \bar{L} = L_{A_t} + L_{Y_t} \quad (5)$$

Current value Hamiltonian is

$$H_t = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta_{1t} [K_t^a (L_t - L_{A_t})^{(1-a)} A_t^{(1-a)} - C_t] + \theta_{2t} b A_t L_{A_t}$$

Two decision variables are C_t and L_{A_t} . The first order conditions are than

$$\frac{dH_t}{dC_t} = \frac{(1-\sigma)C_t^{-\sigma}}{1-\sigma} - \theta_{1t} = 0$$

$$C^{-\sigma} = \theta_1$$

$$g_C = \frac{\dot{C}}{C} = -\frac{1}{\sigma} \frac{\dot{\theta}_1}{\theta_1} = \frac{1}{\sigma} g_{\theta_1} \quad (6)$$

for consumption, and

$$\frac{dH_t}{dL_{A_t}} = 0 \quad (7)$$

$$\theta_1 (1-a)(L - L_A)^{(1-a)} K^a A^{(1-a)} = \theta_2 b A$$

for labor force engaged at R&D sector.

$$\text{Co-state conditions are } \frac{dH_t}{dK_t} = \rho\theta_{1t} - \dot{\theta}_1$$

$$g_{\theta_1} = \frac{\dot{\theta}_1}{\theta_1} = \rho - \frac{a(L - L_A) A^{(1-a)}}{K^{(1-a)}} \quad (8)$$

for physical capital, and

$$\frac{dH_t}{dA_t} = \rho\theta_{2t} - \dot{\theta}_2$$

$$\theta_1(1-a)(L-L_A)^{(1-a)}K^aA^{-a} + \theta_2bL_A = \rho\theta_2 - \dot{\theta}_2 \quad (9)$$

for capital of knowledge.

Solution of the model for balanced growth path gives same constant growth rates for K , Y , and C (given by growth rate of A), and the current value prices θ_1 and θ_2 that decline at constant rate.

From equations (7) and (9) it is possible to derive

$$\frac{\theta_2b(L-L_A)}{A} + \theta_2bL_A = \rho\theta_2 - \dot{\theta}_2 \quad (10)$$

$$\frac{\dot{\theta}_2}{\theta_2} = \rho - bL$$

Now, log differentiating of (7) gives

$$\frac{\dot{\theta}_1}{\theta_1} + a\frac{\dot{K}}{K} + (1-a)\frac{\dot{A}}{A} = \frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{A}}{A} \quad (11)$$

$$\frac{\dot{\theta}_1}{\theta_1} = \frac{\dot{\theta}_2}{\theta_2}$$

Now, from equations (6), (10), and (11) it is possible to obtain equilibrium growth path for this centralized problem

$$g = \frac{\dot{C}}{C} = -\frac{1}{\sigma}\frac{\dot{\theta}_1}{\theta_1} = -\frac{1}{\sigma}\frac{\dot{\theta}_2}{\theta_2} \quad (12)$$

$$g_{SO} = \frac{bL - \rho}{\sigma}$$

where g_{SO} presents socially optimal rate of growth. This rate of growth is different from one we get in decentralized environment, $g = \frac{ab\bar{L} - \rho}{a + \sigma}$, already derived in the main body of this paper.