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Reflexive Game Theory Approach to Mutual Insurance Problem

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ABSTRACT

This paper deals with application of game theory model to insurance market. We observed a form of mutual insurance in conditions of full and partial information obtained by insurance buyers. First part of the paper defines the model of non-cooperative game, followed by principles of optimality, types of stability and equilibrium. Here, model of non-cooperative game has been analyzed under the assumption that players are fully informed. Situation where players are only partially informed requires model which takes into account decision process and analysis of every player's actions. This analysis requires that hierarchical structure among players need to be established. Model of conflict situation, with established hierarchical structure with informational reflexivity, represent the model of reflexive game. We define that reflexive game model, followed by required conditions for which some strategy is informational equilibrium strategy. We then proceed to perform analysis of game theoretic application to mutual insurance model, with hierarchical structure in agent positioning, in conditions of both full and partial information obtained by the player, with the goal of finding equilibrium strategies.

INTRODUCTION

Development of game theory provided lots of opportunity to more accurately model variety of economic ***phenomena*** in observed system. First introduction of game theory to economic literature was trough papers by Cournot (1838), Bertrand (1883), and Edgeworth (1897). John von Neumann and Oscar Morgenstern (1944) in their „Theory of Games and Economic Behavior" established the foundations of general game theory and confirmed possibility to analyze economic ***phenomena*** with game theory tools. John F. Nash (1950) introduced a concept of equilibrium outcome (equilibrium situation) as a method of resolving non-cooperative games.

Basic division of games in accordance to game theory is separation on cooperative and non-cooperative games. Theory of non-cooperative games starts with an assumption that the decision maker (player) is a rational individual that aspires to the best possible outcome (maximizing utility) in accordance with pre-defined rules. Contrary to that, theory of cooperative games starts with the group of players which form a coalition that acts as a decision maker. If the game is well defined, then it is precisely determined what a coalition can achieve, but without indications as to how the outcome can influence the coalition itself (Aumann, 1997). So, theory of non-cooperative games is strategy oriented, and it explores the expected outcomes, or how are the outcomes reached among actions undertaken by players. Contrary to that, theory of cooperative games analyses outcomes directly, without analyzing the way they are achieved. Theory of non-cooperative games represents a micro approach, containing all the details of actions and outcomes, while theory of cooperative games goes for a macro approach, with analysis of all the possible outcomes that can be reached.

We direct attention of further analysis to models of non-cooperative games and aspects of micro analysis, as well as its application to modeling of mutual insurance. Theory of non-cooperative games is a way to model and analyze situations in which optimal strategy for every player is dependent on his own actions, but also dependent on actions taken by other players. Most important characteristic of this theory is built on the fact that players should anticipate how others will act, in accordance to game rules and assumption of rationality. Under those, all players aspire to maximize their utility. During 70's, the focus of analysis was brought to information obtained by rational players as part of non-cooperative games. Important works include Selten (1965) on perfect information and Harsnayi (1967) on partial information. Following those, we have Selten (1975) on concept of equilibrium in extensive games with perfect information, and then Kreps and Wilson (1982) who expanded Selten's model with introduction of partial information on player's payouts. Kreps, Milgrom, Roberts, and Wilson in their experiment of finite game of repetition came to conclusion that players deviate from dominant strategies in some periods of the game, and they aspire to certain level of cooperation. Due to that they analyze repetition model with partial information.

Economic models that regard insurance markets published since early 60's (Borch, 1962 and Bühlmann, 1984) analyze interests of both sides: insurance sellers (insurance companies) and buyers. These models explore the issues with fairness, or Pareto optimality and market equilibrium. Bühlmann claims that insurance premiums depend not only on implied risks, but on market conditions as well. With that, standard actuary tools fail to account for that sort of dependency, and in order to fully include all contributing market interactions we need to develop new models for pricing insurance premiums. In addition to models mentioned, game theory models are being introduced, especially in situations when insurance companies include sub-additive insurance premiums for independent risks. In such cases, individual players can protect their premiums by partaking in mutual insurance, rather than individual. These particular situations are being analyzed by Alegre and Claramunt (1995).

Rothschild and Stiglitz (1976), in their seminal paper, introduce asymmetric information model where every insurant has private information about his state of nature. Insurer provides different contracts, thus providing screening of insurants whit unique separating equilibrium in which high risk insurants chose high risk contracts and vice versa. Picard (2009) further developed R-S model where insurer participate in an insurance as a form of mutual insurance. Moral hazard and adverse selection play important role in insurance and there are number of papers devoted to this area (Geoffard, Chiappori and Durand, 1999; Wambach, 2000; Mimramna and Wambach, 2010). Power and Shubik (1998, 2006) analyze the effect of insurants number and optimal number of reinsured agents, whereas Taksak and Zeng (2011) analyze disproportionate reinsurance.

1. NON-COOPERATIVE GAMES AND OPTIMALITY PRINCIPLE

Game theory models interaction between agents, with the main goal of finding the optimal set of rules (in a form of strategies) for each player. Interaction can be seen as some sort of a conflict in a market, where we have to define who the participants are and which are their possible moves in order to acquire their goal. Every subject in a conflict situation has a set of possible choices - strategies, i.e. his moves in some interaction, or conflict situation that is modeled. Strategies define course of action for each subject in a game.

Depending on used strategies of players the game resolves with some value for each player, as a final "effect" of the interaction. In this setting, therefore, players cannot observe only their strategies, and maximize utility (like in a decision theory setting), and they have to take into account the effect of interaction among different agents. The class of games in which agents do not form coalitions are called non-cooperative games.

Let I be the set of all players. Every player $i \in I$ has a defined strategy set denoted as X_i. Strategy profile represents Cartesian product of strategy sets for every player

$$X = \prod_{i \in I} X_i \ , \tag{1}$$

while with $H_i : X \to$ ˇ we have a defined set of payoff functions for each player, assigning value to each combination of chosen strategies $x \in X$ from a strategy profile set. Normal form non-cooperative game is a defined as a system

$$G = \left\langle I, \{X_i\}_{i \in I}, \{H_i\}_{i \in I} \right\rangle. \tag{2}$$

Suppose that *J* is a subset of players, $J \subseteq I.$ With x_J we will denote subset of strategy space where all strategies of players $j \in J$ are defined. Let $x \| x_J$ be the subset of all strategy profiles where players from subset *J* choose their strategies from set X_J while the strategies of players from subset $I \setminus J$ are taken from $X \setminus X_J$

Strategy profile $x \in X$ dominates strategy profile $y \in X$ if the following condition is satisfied:

$$H_i(y) \le H_i(x), \text{ for all } i \in I, \tag{3}$$

while strict dominance is satisfied if

$$H_i(y) < H_i(x), \text{ for all } i \in I. \tag{4}$$

Definition 1

For a non-cooperative game G strategy profile $x^* \in X$ is I-stabile if there is not any strategy profile $y \in X$ which dominates x^*.

For a I-stabile strategy profile x^* the following statements stand:

(*i*) For every player $j \in J,$ and for each strategy profile $x \in X$

$$H_j(x^* \| x_J) \le H_j(x^*). \tag{5}$$

(*ii*) There exists $j \in J$ for which

$$H_j(x^* \| x_J) < H_j(x^*). \tag{6}$$

Strategy profile satisfying (5) and (6) is called Pareto optimal. Also, directly from definition of I-stabile profiles, if we have a strategy profile $x \in X$ for which $H_j(x^* \| x_J) > H_j(x^*)$ for some player $j \in J$, then there exists player $k \in I$, such that $H_k(x^* \| x_J) < H_k(x^*)$. This conditions show that enhancing one player's payoff function we need to lower the payoff for some other player.

Another optimality principle for non-cooperative games, other than stability principle, is the equilibrium concept.

Definition 2

For a non-cooperative game G, strategy profile $x^* \in X$ is called I-equilibrium if it does not exist any strategy profile $y \in X$ which strictly dominates x^*.

Any change in strategy for each player cannot provide better outcome than the one given by the achieved I-equilibrium. Nash equilibrium is achieved if for all players $i \in I$ and any strategy $x_i \in X_i$

$$H_i(x^* \| x_i) \le H_i(x^*). \tag{7}$$

Non-cooperative game theory models the situations in which the optimal solution for every player is conditional on other players' actions. The anticipation of other players' actions is imposed in game theoretic concept, with assumption that each player is rational and that there exist common knowledge about the rationality of all players, while all the players are maximizing their payoff functions.

Often, normal form game is not good enough to efficiently model economic interaction. Extensive form games are often used to model dynamic interaction between players. The extension form games are represented by player's action in a form of a tree. The elements of an extensive game are (Fudenberg and Tirole, 1991):

(1) The set of players;

(2) The order of moves, i.e. who moves when;

(3) What the players' choices are when they move;

(4) The players' payoff as a function of the moves that were made;

(5) What each player knows when he makes his choices;

(6) The probability distribution over any exogenous events;

Extension form game is defined as a system:

$$G = \left\langle I, T, A, \{X_i\}_{i\in I}, \{H_i\}_{i\in I}, \{N_i\}_{i\in I} \right\rangle. \tag{8}$$

Informational structure of extensive form games provides partition of games into games with perfect and imperfect information. If each information set consists of just one node we consider that game to be a game with perfect information, while in other case we talk about the games with imperfect information. Strategy of a player then represents the description of possible actions depending in which information set he is located.

Definition 3

Let N_i be the family of all information sets of a player *i*, and let *A* be the set of all possible actions in a game *G*. Denote with $C(N) \subset A$ set of all actions in an information set *N*. Strategy x_i of a player *i* is a mapping $x_i : \mathrm{N}_i \to A$ such that for every $N \in \mathrm{N}_i$ follows $x_i(N) \in C(N)$.

Strategy of a player can be viewed as a complete course of action for each of his information sets. It can be seen in a following manner - for each information set where they have a move, all the players provide a decision about their actions, and then we can suppose a computer takes into account these strategies and resolves the game. Strategies therefore represent a complete plan and define behavior of players for all information sets.

Extensive form games with perfect information defined with (8) can be transformed in normal form games where one node represents one possible strategy of a player in a normal form non-cooperative game.

Hierarchical games represent important class of extensive form games. This class is used to model situations where we have interaction between agents (as we mentioned which can be seen as a form of a conflict) with governing power of some player during this interaction. Two-stage governing system of interaction can be defined as follows:

1. Governing (coordination) center I_0 - player with highest governing power (or player with first level governing power);

2. Players $I_i, i = 1,...,n$ with lower governing power (players with second level governing power);

3. Governing (coordination) center I_0 is located at the first hierarchical level and chooses vector $x_0 = (x_1,...,x_i,...,x_n)$ from the set X_0 of control variables, where x_i represents governing action of coordination center toward players with lower governing power.

4. Lower governing power player $I_i, i = 1,...,n$ chooses his action $x_{0i} \in X_i(x_i)$ where $X_i(x_i)$ represents the set of governing action of a player *i*, predefined by governing action x_i of coordination center I_0;

5. The goal of coordination center I_0 is maximization of the function $H_0(x_0, x_{01},...,x_{0n})$ with respect to x_0.

6. Lower governing power players want to maximize function $H_i(x_i, x_{0i})$ with respect to x_{0i};

7. Coordination center has a first move and is able to restrict actions of other players with lower governing power in order to direct their action toward desired goal.

Two stage governing system model of agents' interaction can be formulated as a non-cooperative normal form game with *n+1* players. Assume that player I_0 chooses vector $x_0 \in X$ defined as

$$X_0 = \left\{ x_0 = (x_1,...,x_i,...,x_n) : x_i \geq 0, x_i \in R^l, i = 1,...,n, \sum_{i=1}^{n} x_i \leq b \right\}, b \geq 0 .$$

Vector x_i represents the allocation of the resource l by a player I_0. This resource allocation enables influence on behavior of player I_i from lower hierarchical level. Player I_i observes the action of the player I_0 from higher hierarchical level and then chooses vector $x_{0i} \in X_i(x_i)$. The set of strategies of player I_i is defined as

$$X_i(x_i) = \left\{ x_{0i} \in R^m : x_{0i} A_i \leq x_{0i} + a_i, x_{0i} \geq 0 \right\} .$$

Strategies of player I_i in a game G are defined by the sets of functions $x_{0i}(\cdot)$ that map vector $x_{0i}(x_i) \in X_i(x_i)$ to every element $x_i : (x_1,...,x_i,...,x_n) \in X$. Sets of functions $x_{0i}(\cdot)$ will be from now denoted as X_i for $i=1,...,n.$ Vector x_{0i} can be interpreted as a production plan of a player *i* for a different level of productions, A_i - production matrix of a player *i*, a_i- vector of available resources of a player *i*. Payoff functions are defined in a following manner:

1. Payoff function H_0 of a player I_0 is

$$H_0\left(x_0, x_{01}(\cdot), ..., x_{0n}(\cdot)\right) = \sum_{i=1}^{n} c_i \cdot x_{0i}\left(x_i\right),$$

where $c_i \geq 0, c_i \in R^m$ is a vector with defined coordinates.

2. Payoff function H_i for player $I_i, i=1,...,n$ is denoted as

$$H_i\left(x_0, x_{01}(\cdot), ..., x_{0n}(\cdot)\right) = c_{0i} \cdot x_{0i}\left(x_i\right),$$

where $c_{0i} \geq 0, c_{0i} \in R^m$ is a vector with defined coordinates.

Two stage governing system model of agents' interaction is a normal form game

$$G = \left\langle I_0, \{I_i\}, X_0, \{X_i\}, H_0, \{H_i\}, i=1,...,n \right\rangle. \tag{9}$$

In a game defined by (9) Nash equilibrium exists if following conditions are satisfied:

1. $x_{0i}^*\left(x_i\right) \in X_i\left(x_i\right)$ is the solution of parametric linear programming problem (parameter being vector x_i)

$$\text{for all } i=1,...,n, \quad \max_{x_{0i} \in X_i(x_i)} c_{0i} \cdot x_{0i} = c_{0i} \cdot x_{0i}^*\left(x_i\right), \tag{10}$$

2. $x_0^* = \left(x_1^*, ..., x_n^*\right) \in X_0$ is the solution of the nonlinear programming problem

$$\max_{x_0 \in X_0} H_0\left(x_0, x_{01}^*(\cdot), ..., x_{0n}^*(\cdot)\right) \tag{11}$$

We need to show that $\left(x_0^*, x_{01}^*(\cdot), ..., x_{0n}^*(\cdot)\right)$ is Nash equilibrium. Based on (11) we have that for a player I_0 with highest governing power optimal solution is provided if

$$\forall x_0 \in X_0, \; H_0\left(x_0^*, x_{01}^*(\cdot), ..., x_{0n}^*(\cdot)\right) \geq H_0\left(x_0, x_{01}^*(\cdot), ..., x_{0n}^*(\cdot)\right).$$

Based on (10) we have that for all players I_i, $i=1,...,n$ with lower governing power optimal strategy is the one satisfying that for all $x_{0i}(\cdot) \in X_i$

$$H_i\left(x_0^*, x_{01}^*(\cdot), ..., x_{0n}^*(\cdot)\right) \geq H_i\left(x_0, x_{01}^*(\cdot), ..., x_{0i-1}^*(\cdot), x_i(\cdot), x_{0i+1}^*(\cdot), ..., x_{0n}^*(\cdot)\right).$$

Therefore, $\left(x_0^*, x_{01}^*(\cdot), ..., x_{0n}^*(\cdot)\right)$ is Nash equilibrium. Equilibrium $\left(x_0^*, x_{01}^*(\cdot), ..., x_{0n}^*(\cdot)\right)$ is I-stabile that is, enlarging the payoff value of one player with result with lower payoff value for other.

2. MODEL OF REFLEXIVE GAMES

Reflexive models are applicable in situation where we do not have knowledge about the level of players' information, and where players define their actions according to hierarchical structure imposed. In order to incorporate reflexive models in game theory it is required to define strategic and informational reflexivity. Strategic reflexivity represents the reasoning and the results of reasoning process of players regarding the principles and strategies of other players within the scope of their information set. At the same, players this process accredit as the result of strategic reflexivity. Therefore, strategic reflexivity is a process where we anticipate other players' action from possible set of actions. Informational reflexivity assumes reasoning process of players regarding principles and strategies of other players, but, unlike in strategic reflexivity does not impose actions based on this reasoning. Informational reflexivity is in correlation with a situation where there is lack of mutual information and their results are used together with strategic reflexivity (Novikov and Chkhartishvili, 2003a; Novikov and Chkhartishvili, 2003b).

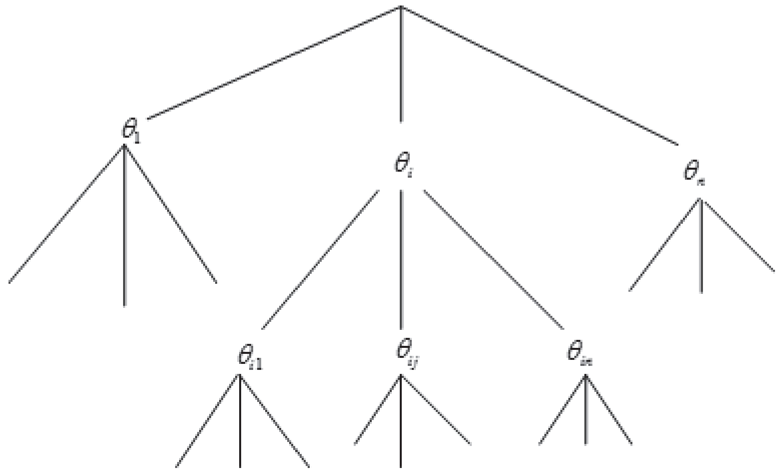
This level of reasoning requires hierarchical structure setting among players. This has been modeled with a class of extensive form dynamical games of hierarchical structure. Two stage governing system model of agents' interaction is defined and we showed that there exists Nash equilibrium in this setting.

Games of interaction between agents with a defined hierarchical structure between players with existing informational reflexivity are called reflexive games (Novikov and Chkhartishvili, (003a; Novikov, and Chkhartishvili, 2003b). Psychological research about reflexive models by Lefebvre (1965) represent foundation for reflexive game models (see also Lefebrve, 2010) for detailed concept of reflexive games and their methodological background).

In a reflexive game setting every player models and anticipates other players' moves based on available information, i.e. every player chooses his action simulating interaction with a player from higher governing power level, and waiting for a same governing level players to choose their actions. Steady state, based on resulted strategies in a reflexive game setting represents *informational equilibrium*. Therefore, equilibrium in any reflexive game is dependent on an informational structure. Moreover, the change of informational structure leads to the change of an informational equilibrium.

Let $I = \{1,2,...,n\}$ be the set of players, while $N = (N_1, N_2, ..., N_n)$ is the informational set of all players from set I. Every player *i* has an compact and convex information set, $N_i = (\theta_i, \theta_{ij}, \theta_{ijk}, ...)$, for $i, j, k \in I$ and we say that N_i is the information structure of player *i*. Elements of the set N_i, where $(\theta_i, \theta_{ij}, \theta_{ijk}, ...) \in N_i$ represents the states of the nature, i.e. θ_i is the information player *i* has about the state of the nature. Element θ_{ij} is the information player *i* has about the "informational knowledge" of player *j*. Let Ξ be the set of all finite index sequences from set *I* , and let σ be some index sequence, that is element of set Ξ, and $|\sigma|$ his cardinality, or the number of element in that sequence. By choosing action x_i player i has appointed information set $N_i = (\theta_i, \theta_{ij}, \theta_{ijk}, ...)$ after which is possible to model his strategy. After deciding strategy for player *i*, strategies of other players are defined. Therefore, defining the value of the game requires taking into consideration strategies of all players regardless of his hierarchical level. Information structure of player i can be represented in a form of a tree with nodes θ_i, as shown in Figure 1.

Figure 1. Information structure of a player



Reflexive game can be represented as a system

$$G=\left\langle I,T,A,\{X_i\}_{i\in I},\{H_i\}_{i\in I},N\right\rangle, \tag{12}$$

where

- Elements of set *I* are players
- Final set of nodes T
- Final set of moves A
- Convex and compact set of strategies X_i
- The game resolves as a consequence of combination of strategies provided by players

$$X=\prod_{i\in I}X_i\ , \tag{13}$$

Continuus and convex payoff function H_i for a player i after the realization of the game

$$H_i : X \to R\,, \tag{14}$$

- Set $N=(N_1,N_2,...,N_n)$ represents all the information sets for all players.

Formal definition of reflexive games allows formalizing concept of informational equilibrium (Novikov, Chkhartishvili, 2003a):

Definition 4

For all $\tau\in\Xi$ strategy x_τ^* is informational equilibrium if following conditions are satisfied

- Structure of the set N is finite
- For all $\lambda,\eta\in\Xi$, $N_{\lambda i}=N_{\eta i}\Rightarrow x_{\lambda i}^*=x_{\eta i}^*$
- For all players $i\in I$ and for a finite index sequence $\sigma\in\Xi$ strategy $x_{\sigma i}^*$ satisfies condition

$$x_{\sigma i}^*\in\arg\max_{x_i\in X_i}H_i\left(\theta_{\sigma i},x_{\sigma i,1}^*,...,x_{\sigma i,i-1}^*,x_i,x_{\sigma i,i+1}^*,...,x_{\sigma i,n}^*\right)$$

First condition of the Definition 4 states that reflexive game model requires finite set of players of different hierarchical levels. Second condition states that players with same information sets chose the same strategies. The third one determines rational behavior of the players, where every player maximizes their payoffs while taking into the account actions of other players. Informational vector $(\theta_i, \theta_{ij}, \theta_{ijk}, \ldots)$ of a player *i* is Nash equilibrium if for any strategies $x_i, y_i \in X_i$ following inequality holds:

$$H_i(\theta, x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+1}) \geq H_i(\theta, x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_{i+1}).$$

3. REFLEXIVE GAME MODEL OF A MUTUAL INSURANCE

Model of mutual insurance with a hierarchical structure of agents related to the probability of damage of the insured object is presented. We analyze the relation between insurants while paying premiums, and fund used for claims of insurants. We assume the same risk attitude of insurants (agents) and different probabilities of damage realization and different claim structure of agents.

The main goal of the model is to impose insurants reflexivity and their incomplete information. Identical risk attitude and perfect information would allow arbitrary (a priori defined) reallocation of collected claims between agents. But, adding incomplete information creates asymmetry allowing different payoffs of collectible claims i.e. different expect utilities for different agents.

Let $N = \{1, \ldots, n\}$ be the set of insurant (or agents) who have signed a contract with an insurer. Every agent has an expected utility function

$$f_i = \pi_i - r_i + p_i(h_i - q_i) \quad , \tag{15}$$

where

π_i - profit of agent *i* after his economic activity (for example salary)

r_i - insurance premium of agent *i*

p_i - probability of damage to the insured object, independent among agents

h_i - collectible claim of agent *i*

q_i - loss after the damage of insured object

In a perfect information setting, overall insurance premium is equal to

$$R = \sum_{i=1}^{n} r_i \, , \tag{16}$$

while overall expected claims that will be collected, denoted as *H* are therefore

$$H = \sum_{i=1}^{n} p_i h_i \, . \tag{17}$$

With perfect information of all agents we have allocation of premiums toward the clients who had to collect the insurance, that is

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} p_i h_i \, , \tag{18}$$

which is the equilibrium condition in a mutual insurance setting. This condition can be viewed as a zero-sum game model, as in our model insurance company works with zero profit margin and all the premiums are transferred to agents who have suffered damage of the insured object. Also, assume complete compensation of the damage,

$$h_i = q_i . \tag{19}$$

With assumption (19) directly follows that expected value of claims *H* is equal to the expected value of the incurred damage:

$$H = \sum_{i=1}^{n} p_i q_i . \tag{20}$$

With imposed rationality and directly from (18) and (20) we see that an agent *i* will be ready to pay a premium equal to

$$r_i = p_i q_i . \tag{21}$$

In order to have equality (21) satisfied, or to have every agent premium equal to the expected value of damage following two conditions need to be satisfied:

- All individual insurance parameters need to be common knowledge to all agents;
- All agents behave according to their own parameter values.

Failure of one of these conditions would imply that a rational agent cannot determine his premium as an expected value of the damage and equation (21) does not hold. Of course, these assumptions are far from realistic. What we need to take into the account is the information that insurants have about the probability of damage. What a potential insurant might represent as a probability might differ from the true value. Mechanism of control must be induced to take care about the validity of the presented parameters. Assume that agents can share their information about the probability p_i. Every insurant following (21) has interest to lower this probability and therefore will provide information that his probability is s_i instead of the true value. This information is provided to all agents. Directly we have

$$r_i(s_i) = s_i q_i . \tag{22}$$

Collected premiums are therefore equal to

$$R(s) = \sum_{i=1}^{n} r_i(s) = \sum_{i=1}^{n} s_i q_i . \tag{23}$$

If the damage of the insured good happens, the claim that should be collected, $h_i(s)$, can be defined commensurably to the collected premiums, meaning that for all $i = 1,...,n$,

$$h_i(s) = \alpha(s) q_i , \tag{24}$$

where $\alpha(s)$ represents the coefficient of mutual participation in the overall sum of collected premiums, as a ratio between expected value of the collectible claim and overall sum of the premiums. The choice of $\alpha(s)$ represent the set of all strategies for all agents in a mutual insurance setting.

Expected utility function of an agent *i* can now be represented as

$$f_i = \pi_i - r_i(s_i) + p_i(h_i(s) - q_i) . \tag{25}$$

Using (22) and (24) we obtain

$$f_i = \pi_i - s_i q_i + p_i\left(\left(\alpha(s) q_i - q_i\right)\right),$$

from which directly follows

$$f_i = \pi_i - s_i q_i + p_i \alpha(s) q_i - p_i q_i \,. \tag{26}$$

From (26) we see that $s_i q_i$ represents insurance premium while the claim of an agent is given with an expression $p_i \alpha(s) q_i$. Rationality of agents implies that every agent wants to pay smaller premium than potential claim that can be collected - $s_i q_i \le p_i \alpha(s) q_i$. Directly

$$s_i \le p_i \alpha(s) \,. \tag{27}$$

Strategy of an agent given by the choice of coefficient $\alpha(s)$ can be defined as

$$\alpha(s) = \min\left\{\frac{R(s)}{H}, 1\right\}. \tag{28}$$

For a mutual insurance with perfect information the equilibrium condition is given by equation (18), directly implying $\alpha(s) = 1$. Based on (27) for all players we have $s_i \le p_i$. Directly follows that $R(s) < H$ which would change the existing equilibrium. In order to overcome described perturbation of the equilibrium we can induce following mechanism:

- At the beginning of the insured period each player has to announce his probability of the damage occurrence;
- At the end of period overall sum of claims should be compensated due to difference of probabilities p_i and s_i;
- the total amount of compensated sum is determined by a probabilities s_i.

In equilibrium overall premiums should be equal to the expected values of claims:

$$\sum_{i=1}^{n} r_i(s) = H \,, \tag{29}$$

where the strategies of player i are the choice of $r_i(s)$. Given (17) expected utility function of an agent is

$$f_i = \pi_i - r_i(s) \,. \tag{30}$$

As expected value of premiums is equal to expected loss of the insured object, every agent wants to set a premium of insurance to be less than expected loss of the insured object:

$$r_i(s) \le p_i q_i \,. \tag{31}$$

Defining strategy that sets every agents' premium proportional to the level of expected loss

$$r_i(s) = \frac{s_i q_i}{\sum_{i=1}^{n} s_i q_i} H \,, \tag{32}$$

then expected utility function (30) is optimal when the risk premium $r_i(s)$ is minimized. Strategy of an agent defined with (31) and (32) still can lead to the disruption of the equilibrium condition. Incorrect information about the probability can significantly reduce insurance premium. To reduce the incentive of providing false probability strategy should take into account number of agents in which the size of premium is lowered with greater number of agents. Example of such strategy can be in a form:

$$r_i(s) = \frac{\frac{1}{s_i}}{\sum_{i=1}^{n} \frac{1}{s_i}} H \quad . \tag{33}$$

Changing strategy values in the expected utility function (30) we obtain

$$f_i = \pi_i - \frac{\frac{1}{s_i}}{\sum_{i=1}^{n} \frac{1}{s_i}} H \, . \tag{34}$$

Strategy defined with (33) still does not impose credibility to agents, and only provide additional incentive of agents to include new premiums.

3.1 Extension of the model

So far, in a described model, we showed that agents have an incentive to reduce the actual probability p_i thus creating a setting in which Nash equilibrium of game with perfect information cannot be obtained. Introducing new aspects of the model did not manage to create credibility to their assumptions about the probability, but only create incentive to include more agents in the mutual insurance.

We introduce coordination center as a player of higher hierarchical level. We assume that coordination center has additional information about the insurant and henceforth we continue to model this as a reflexive game. Assume that coordination center acquired information about the potential losses of the agent *i* if insured object is damaged and let that be denoted as q_i^c, while the information about the probability of damage is p_i^c. This does not have to be the same as actual values of the probability p_i and value of the damage q_i. At the beginning, every agent informs coordination center whether he will join the mutual insurance process. If player decides to join the process he has to announce to coordination center his probability of claim s_i in a form of monetary value of their premium $r_i(s) = s_i q_i$. Once paid premium cannot be retracted. After each player announces their probabilities and losses, coordination center is checking is the probability of claim given by players bigger of the assumption made by coordination center:

$$\sum_{i=1}^{n} r_i(s) \geq \sum_{i=1}^{n} p_i^c q_i^c . \tag{35}$$

This would imply that projected probability of claim realization, given as a monetary value $r_i(s)$ that players provide should be at least as big as the expected value of future losses, by calculation of coordination center. Coordination center based on the assumption that the expected loss $\sum_{i=1}^{n} p_i^c q_i^c$ should be equal to the expected sum of claims $H^c = \sum_{i=1}^{n} p_i^c h_i^c$ forces players from lower hierarchical levels to sign the contract. Expected utility function of agent *i* is then equal to

$$f_i = \pi_i - r_i(s) + p_i^c\left(h_i^c - q_i^c\right). \qquad (36)$$

As we know that coordination center serves as guarantor, for each agent $p_i^c h_i^c = p_i^c q_i^c$, transforming expected utility function in a form

$$f_i = \pi_i - r_i(s). \qquad (37)$$

Rationality of players impose the fact that the risk premium is at most equal to expected damage incurred, $r_i(s) \le p_i q_i$. Players' optimal strategy is premium, proportional to the incurred damage

$$r_i(s) = \frac{s_i q_i}{\sum_{i=1}^{n} s_i q_i} H^c. \qquad (38)$$

Maximum of the expected utility function (37) is reached when function $r_i(s)$ obtains its minimal value. Defined strategy eliminates adverse selection problem as coordination center provides control about the probabilities of damage, which prevents unrealistic change of insurance premiums.

Premium given as (38) satisfies condition (35) of coordination center maximizes utility function of each agent, which represents the Nash equilibrium. Unlike D. Novikov, and A. Chkhartishvili (2004) where their model requires that all players have to enter mutual insurance in order to provide equilibrium, we defined a model where this is not required. Whereas one player in their model leaving the mutual insurance contract would cause collapse of the fund and all players leaving the fund, our model does not have such a restrictive setting and is far more realistic.

CONCLUSION

We observed mutual insurance model in conditions of complete information, and in condition of partial information obtained by insurance buyer. With complete information, in order for equilibrium to exist, total insurance premium needs to be equal to total expected damage compensation. Established equilibrium condition of mutual insurance model is observed as zero-sum game model condition. With that, in order to satisfy equilibrium condition, all players need to be completely informed of insurance parameters, and all players need to conform to those parameters. The biggest problem in this model is to force players to behave according to their parameters. Having all the probability of the model, we could see this as a decision theory problem. Realistically, this will never happen.

Partial information requires additional analysis when exploring possibility to reach equilibrium. We introduced coordinating center as a player of higher hierarchical level to mutual insurance model, and it has access to additional information. Such model is further analyzed as a reflexive game model. We assumed that coordination center has additional information about the insurants and the probability of damage occurrence, and based on this information coordination center provide claims when necessary, thus motivating agents to sign the contract of mutual insurance. Condition that needs to be satisfied in order to acquire required sum of money for claims is that overall sum of expected premiums have to be equal to sum of expected losses. Premiums equal to expected losses occurred, assumed by coordination center, represent Nash equilibrium.

Two-stage hierarchical model of mutual insurance represents realistic model of economic interaction. It provides partial information setting of agents, whereas coordination center obtains additional information about all parameters of insured cases. Its actions forces potential agents to sign the contract of mutual insurance as well as to uphold to true values of their parameters. Every agent abandoning the agreement provides greater fond for all the other agents. Furthermore, our model overcomes strict assumptions of similar models related to reflexive game, thus providing much realistic setting.

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