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CURVATURE AND THE ELASTICITY OF SUBSTITUTION: WHAT IS THE LINK?*

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ABSTRACT
Relation between curvature and the elasticity of substitution is the old question important for economic theory. Opinions of economists concerning presence or absence of a link between these two concepts radically diverge. Also now there is a steady trend of the use of the Arrow-Pratt coefficient of relative risk aversion and the coefficient of relative prudence as characteristics of utility functions and production functions even in non-stochastic models, and these two coefficients are also commonly interpreted as measures of curvature. The purpose of the paper is to contribute to clarification of the links between all these concepts. We suggest a simple unifying approach based on the notions of prototype functions and osculating curves. In framework of this approach we easily derive the classic geometric curvature and show the relations between the Arrow-Pratt coefficient, the prudence coefficient, the elasticity and the elasticity of substitution. As an example, demonstrating the role of such relations in economic models, we study a simple macroeconomic model with a non-homothetic production function.

KEYWORDS: *curvature, elasticity of substitution, production function, utility function, CES function, osculating function, Arrow-Pratt coefficient of relative risk aversion, relative prudence coefficient*

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1. INTRODUCTION

Utility functions and production functions are basic research tools both in macroeconomics and microeconomics and, permanently, attempts are being made to develop and connect various concepts to characterize such functions. First of all, these are concepts of curvature and elasticity of substitution. These notions are often used in discussions of economic phenomena depending on parameters of production functions or utility functions.

Question of relationship between elasticity of substitution and curvature proves to be not straightforward. In different situations these two concepts seem to be close to or far from each other in their meaning.

Indeed, for the Cobb-Douglas function with constant returns to scale (CRS), $F(K, L) = AK^{\alpha}L^{1-\alpha}$, where $\alpha \in (0,1)$, elasticity of substitution is identically equal to 1. At the same time, curvature, κ, varies at different points:

$$\kappa = \frac{\alpha(1-\alpha)KL}{[\alpha^2 L^2 + (1-\alpha)^2 K^2]^{\frac{3}{2}}}.$$

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The function $\kappa(K, L)$ is homogeneous of degree (-1).

On the other side, it is easy to provide an example of interdependency of the elasticity of substitution and curvature. In particular, one can check that for the family of the constant elasticity of substitution (CES) functions $(\alpha K^p + (1-\alpha)L^p)^{1/p}$, where $p \in (-\infty, 0) \cup (0,1)$, $\alpha \in (0,1)$, at point $K = 1, L = 1$ the curvature is $\kappa = c(\alpha)/\sigma$, where $\sigma = 1/(1-p)$ is the elasticity of substitution, and $c(\alpha)$ is a constant which depends on parameter α.

Opinions of economists concerning presence or absence of connection between the elasticity of substitution and curvature are radically different.

Among the first who stressed presence of a link between the curvature and the elasticity of substitution was John Hicks who, in particular, wrote: "The curvature of the indifference-curve describes the same property as the "rate of increase of the marginal rate of substitution". But to take either as our measure without correction for units would be impossible – the result would have as little significance as the uncorrected *slope* of a demand curve. A measure free from this objection fortunately now lies ready to our hand. It is the *elasticity of substitution…"* (Hicks, 1934, pp. 58-59).

In today's texts on microeconomics it is also easy to find claims, without arguments, that there is a link between these two notions. Here are but some examples. "The elasticity of substitution measures the *curvature* of the isoquant" (Varian 1992, p. 13); "the *elasticity of substitution*… captures the relationship between the input ratio and the curvature of the isoquants" (Gravelle and Rees 2004, p. 29); "The elasticity of substitution $\sigma_{ij}(x^0)$ is a measure of the curvature of the $i-j$ isoquant through x^0 at x^0" (Jehle and Reny 2011, p. 129).

Contrary, De La Grandville (1997) argues that "there is no link between curvature and the elasticity of substitution".

Probably, the authors of the microeconomic texts have in mind local differences in curvature, such as in our above example with the CES functions, in which the curvatures are compared in a fixed point. Opposite to this, De La Granville speaks about global comparisons. To give one more example, for all CES functions $(\alpha K^p + (1-\alpha)L^p)^{1/p}$ with any α and at all points (*K*, *L*), the elasticity of substitution, under fixed parameter *p* is the same and equal to $\sigma = 1/(1-p)$, while curvature is different even , even under fixed α and even at points belonging the same ray $K/L = const$.

The necessity to explain conceptual links between basic concepts relates not only to the elasticity of substitution and curvature but to some other important relative concepts as well. In economic literature there is now a steady trend to use CRS utility and production functions in their intensive form, which is an increasing strictly concave function of one variable, $f(x)$, and to study models with evoking as tools such functions as

$$r_f(x) = -\frac{f''(x)x}{f'(x)},$$

and

$$r_{f'}(x) = -\frac{f'''(x)x}{f''(x)}.$$

Initially the functions $r_f(x)$ and $r_{f'}(x)$ appeared in risk analysis, where they are called, correspondingly, *the Arrow-Pratt coefficient of relative risk aversion* and *the coefficient of relative prudence* (Pratt 1964; Arrow 1965; Kimball 1990; Maggi et al. 2006; Martins 2007). However, these functions became popular also in analysis of various non-stochastic models in macroeconomics (Blanchard and Fischer 1989, Carroll and Kimball 1996), industrial organization (Zhelobodko et al. 2012; Levine 2012), international trade (Mrázová and Neary 2013), public economics (Andersen and Bhattachary 2013).

Usually when the authors use the functions $r_f(x)$ and $r_{f'}(x)$, they interpret them as measures of curvature; however, the link of these functions with standard mathematical characteristics of the curvature is not disclosed. Moreover, the economic sense of the coefficients r_f and $r_{f'}$ often stays in shadow in situations not related to risk and uncertainty.

The purpose of our article is to contribute to clarification of the link between curvature, elasticity of substitution and such characteristics of utility function and production function as r_f and $r_{f'}$.

We suggest a simple unifying approach to different possible concepts of curvature based on the notions of prototype functions and osculating curves. In framework of this approach, we show that the classic curvature and the elasticity of substitution are, in some sense, "on different shores": these concepts are parallel in their origin, but relate on the use of different geometric curves: circles and graphs of CES functions, correspondently. The classic curvature is obtained when the prototype function defines circles. The functions $r_f(x)$ and $r_{f'}(x)$ arise in natural way when the prototype form is the intensive form of the Cobb-Douglas function or the CES function – correspondently, the elasticity of substitution is constant in these cases and there is a relation of the elasticity of substitution, the Arrow-Pratt coefficient and the prudence coefficient.

We also obtain several previously unknown relations between the Arrow-Pratt and the prudence coefficients, the elasticity of the CES function, and its elasticity of substitution. As an example demonstrating the role of such kind of relations in economic models, we study a simple macroeconomic model with a non-homothetic production function.

The paper is organized in the following way. In Section 2 the notion of the elasticity of substitution is formulated and some of its properties are proved. In Section 3 we introduce general concepts of the prototype form and the osculating functions. By use of these concepts we easily derive the classic geometric curvature. In Section 4 we apply the same approach to the Cobb-Douglas prototype form; this leads to relations between the Arrow-Pratt coefficient, the elasticity and the prudence. In Section 5 in a similar way on basis of the CES prototype form we obtain a link of these variables with the elasticity of substitution. In Section 6 we consider an example of application of our results to a closed economy model with a non-homothetic production function. In section 7 we make a conclusion.

2. CALCULATION OF ELASTICITIES

Let us remind that if *f* and *x* are in functional relationship (possibly, through another variable) then $\varepsilon_f = \frac{df}{dx}\frac{x}{f}$ is called *elasticity* of *f* with respect to *x.* From the properties of differentiation it is immediately clear that the elasticities can be calculated according to very simple rules: elasticity of a product *fg* is equal to the sum of elasticities $\varepsilon_f + \varepsilon_g$; elasticity of a quotient f/g is equal to the difference of elasticities $\varepsilon_f - \varepsilon_g$; elasticity of a power f^α is equal to $\alpha\varepsilon_f$; elasticity of a constant is zero. Finally, multiplying a function by a constant has no effect on its elasticity (this is different from the usual differentiation).

It is obvious that the Arrow-Pratt coefficient, with a minus sign, is equal to the elasticity of the derivative of the function:

$$r_f = -\varepsilon_{f'},$$

and the coefficient of relative prudence is equal, also with a minus sign, to the elasticity of the second derivative:

$$r_{f'} = -\varepsilon_{f''}.$$

LEMMA 1. *a) Elasticity of elasticity, i.e. the quantity* $\varepsilon_{\varepsilon f} = \dfrac{\varepsilon_f' x}{\varepsilon_f}$, *equals* $(-r_f + 1 - \varepsilon_f)$.

b) Elasticity of the Arrow-Pratt coefficient of relative risk aversion, i.e. the quantity $\varepsilon_{rf} = \dfrac{r_f' x}{r_f}$, *equals* $(r_{f'} - 1 - r_f)$.

Proof. See the Appendix.

COROLLARY 1. *The derivatives of the elasticity and of the Arrow-Pratt coefficient of relative risk aversion are*

$$\varepsilon_f' = (-r_f + 1 - \varepsilon_f) \cdot \frac{f'}{f},$$

$$r_f' = (r_f + 1 - r_{f'}) \cdot \frac{f''}{f'}.$$

Let $F(X_1, X_2)$ be continuously differentiable function of two variables on R_+^2. Then the quantity defined by $S_{12} = -\dfrac{dX_1}{dX_2} = \dfrac{\partial F / \partial X_2}{\partial F / \partial X_1}$ is called *marginal rate of substitution*[3]. The latter equality in this equation holds due to the implicit function theorem. Let us examine the ratio $x = \dfrac{X_1}{X_2}$. Elasticity of x with respect to S_{12} is called the *elasticity of substitution*. The standard notation for the elasticity of substitution is σ.

Let us further assume, that the function $F(X_1, X_2)$ is homogeneous of degree γ, i.e. $F(\mu X_1, \mu X_2) = \mu^\gamma F(X_1, X_2)$ for any vector $(X_1, X_2) \in R_+^2$ and for any $\mu > 0$. It is possible to express the homogeneous function as $F(X_1, X_2) = X_2^\gamma f(x)$, where $f(x) = F(x, 1)$, $x = X_1 / X_2$. Function f in this expression is called the *intensive form* and F is called the *extensive form*.

LEMMA 2. If $F(X_1, X_2)$ *is homogeneous of degree* γ, *then the marginal rate of substitution is equal to*

$$S_{12} = x\left(\frac{\gamma}{\varepsilon_f} - 1\right). \quad (1)$$

Proof. See the Appendix.

PROPOSITION 1. If $F(X_1, X_2)$ *is homogeneous of degree* γ, *then the elasticity of substitution is equal to*

$$\sigma = \frac{\gamma - \varepsilon_f}{(1-\gamma)\varepsilon_f + \gamma r_f}.$$

Proof. See the Appendix

COROLLARY 2. If $\gamma = 1$, *i.e. function f is linearly homogeneous (CRS), then*

$$\sigma = \frac{1 - \varepsilon_f}{r_f}. \quad (2)$$

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[3] Obviously, $S_{12} = S_{21} = 1$

Equation (2) connects the Arrow-Pratt coefficient, r_f, the elasticity of the intensive form, ε_f, while the elasticity of factor substitution of the extensive form, σ.

Thus, the inequality $1-\varepsilon_f - r_f < 0$ corresponds to the case of low substitutability for the extensive form function ($\sigma < 1$), and the inequality $1-\varepsilon_f - r_f > 0$ corresponds to the case of high substitutability ($\sigma > 1$).

2. PROTOTYPE FORMS, OSCULATING FUNCTIONS, AND THE CURVATURE

Let function $f(x)$ be defined in a neighborhood of point $x_0 \in R_+$, *n* times continuously differentiable, increasing and strictly concave:

$$f'(.) > 0,\ f''(.) < 0\,. \quad (3)$$

It is important to note that conditions (3) introduce limitations on elasticity, $\varepsilon_f(x)$.

LEMMA 3. *If a continuous function $f(.)$ is defined on R_+, satisfies (3), and $f(0) \geq 0$, then function $f(.)$ is inelastic, i.e. $\varepsilon_f(.) < 1$.*

Proof: See the Appendix.

It is easily seen that the inelasticity of $f(.)$ is equivalent to decrease in the ratio $f(x)/x$.

Let us assume that the form of the function $f(x)$ is unknown, however, at point x_0 the values of the function and its *n* derivatives are given, $f(x_0),\ f'(x_0),\ldots,\ f^{(n)}(x_0)$. Or, what is more common for economic models, one can know the value $f(x_0)$ and some economic measures, such as $\varepsilon_f(x_0)$, $r_f(x_0)$, $r_{f'}(x_0)$. This information is equivalent, since these measures are, in their turn, based on the values of the function and its derivatives.

Let $g(x,\gamma_1,\ldots,\gamma_k)$ be a class of functions with k parameters; this class will be referred as *prototype form*. Assume that the values of parameters $\bar\gamma_1,\ldots,\bar\gamma_{n+1}$ can be found from the system of equations:

$$g(x_0,\bar\gamma_1,\ldots,\bar\gamma_{n+1}) = f(x_0),\ g^{(i)}(x_0,\bar\gamma_1,\ldots,\bar\gamma_{n+1}) = f^{(i)}(x_0),\ i = 1,\ldots,n\,, \quad (4)$$

where $g^{(i)}, f^{(i)}$ are the *i*-th derivatives. In such case we say that functions $g(x,\bar\gamma_1,\ldots,\bar\gamma_{n+1})$ and $f(x)$ are *osculating* at the point x_0 and, correspondingly, their graphs are *osculating curves*.

In the classic geometric case, function $f(x)$ at point x_0 is characterized by the value $f(x_0)$ and by two derivatives, $f'(x_0)$ and $f''(x_0)$. A two-parameter prototype form $y = g(x, a, b)$ is determined as implicit function by the equation of circle with center at a point (a, b). It is clear that the osculating circle for $f(x)$ at point x_0 satisfies the following system of equations:

$$\begin{cases} (x-a)^2 + (y-b)^2 = (x_0-a)^2 + (f(x_0)-b)^2, \\ \left.\dfrac{dy}{dx}\right|_{x_0} = f'(x_0), \\ \left.\dfrac{d^2y}{dx^2}\right|_{x_0} = f''(x_0). \end{cases}$$

Evidently, this system reduces to

$$\begin{cases} -\dfrac{x_0-a}{f(x_0)-b}=f'(x_0), \\ -\dfrac{(f(x_0)-b)^2+(x_0-a)^2}{(f(x_0)-b)^3}=f''(x_0). \end{cases}$$

From this system, without explicit solving for parameters *a*, *b*, it is easy to obtain the expression for the radius of the osculating circle:

$$R=[(x_0-a)^2+(f(x_0)-b)^2]^{1/2}=\frac{[1+(f'(x_0))^2]^{3/2}}{|f''(x_0)|}.$$

The value R is the *radius of curvature*; and the inverse value, $\kappa=1/R$, is the *curvature* of function $f(x)$ at point x_0.

With regard to production functions and utility functions possessing CRS, the calculation of the curvature is naturally done on the basis of extensive form. Since the curvature has no CRS, transition to intensive form would give no advantage.

As an example, we will derive the formula mentioned in Section 1. It is the formula for the curvature of an isoquant of the Cobb-Douglas production function $F(K,L)=AK^{\alpha}L^{1-\alpha}$, where $A>0,\ \alpha\in(0,1)$. On an isoquant: $K^{\alpha}L^{1-\alpha}=C=const.$ We find

$$L=C^{\frac{1}{1-\alpha}}K^{\frac{\alpha}{\alpha-1}},\ L'=C^{\frac{1}{1-\alpha}}\cdot\frac{\alpha}{\alpha-1}\cdot K^{\frac{1}{\alpha-1}},\ L''=C^{\frac{1}{1-\alpha}}\cdot\frac{\alpha}{(\alpha-1)^2}\cdot K^{\frac{2-\alpha}{\alpha-1}}.$$

Substituting *C* for $K^{\alpha}L^{1-\alpha}$, we get

$$L'=\frac{\alpha}{\alpha-1}\cdot\frac{L}{K},\ L''=\frac{\alpha}{(\alpha-1)^2}\cdot\frac{L}{K^2}.$$

The curvature at a point (K,L) equals

$$\kappa=\frac{|L''|}{(1+(L')^2)^{\frac{3}{2}}}=\frac{\frac{\alpha}{(1-\alpha)^2}\cdot\frac{L}{K^2}}{\left(1+\frac{\alpha^2}{(1-\alpha)^2}\cdot\frac{L^2}{K^2}\right)^{\frac{3}{2}}}=\frac{\alpha(1-\alpha)KL}{[\alpha^2L^2+(1-\alpha)^2K^2]^{\frac{3}{2}}}.$$

3. THE USAGE OF THE COBB-DOUGLAS PROTOTYPE FORM

Now let us consider a function in its intensive form, $f(x)$. Let us use as prototype form the power function $g(x,A,\alpha)=Ax^{\alpha}$ with two parameters, $A>0,\ 0<\alpha<1$. Recall that this functional form is widely used in economics as the intensive form of the Cobb-Douglas function[4]. Given $f(x_0),\ f'(x_0)$, system (4) turns into the following system of equations with unknowns A and α:

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[4] The function Ax^{α} is often referred as CRRA (constant relative risk aversion) function.

$$\begin{cases} Ax^{\alpha} = f(x_0) \\ A\alpha x^{\alpha-1} = f'(x_0) \end{cases}.$$

Solving this system, we find $\alpha = \varepsilon_f(x_0)$, $A = f(x_0)x_0^{-\varepsilon_f(x_0)}$. Hence the osculating function generated by the prototype form *g* is

$$y = f(x_0)\left(\frac{x}{x_0}\right)^{\varepsilon_f(x_0)}. \quad (5)$$

Condition $\alpha < 1$ implies the inelasticity: $\varepsilon_f(x_0) < 1$[5].

THEOREM 1. *Three following statements are equivalent:*

1) *The curvature of the osculation function (5) at point x_0 is higher (lower) than the function's f curvature;*
2) *Elasticity ε_f is increasing (decreasing) at point x_0;*
3) *At point x_0 the following inequality holds, which links the elasticity and the Arrow-Pratt coefficient:*

$$1 - r_f - \varepsilon_f > (<) 0. \quad (6)$$

Proof: See the Appendix.

Conditions like (6) recently appeared in economic literature. E.g., Mrázová and Neary (2013), when analyzing behavior of firms, introduce a notion of so called superconvex function which is defined in a similar way to $1 - r_f - \varepsilon_f \leq 0$. Levine (2012) in a framework of analysis of production chains defines a so called moderately concave function as satisfying the inequality similar to $1 - r_f - \varepsilon_f \geq 0$.

In case of coinciding curvatures (or second derivatives, or Arrow-Pratt coefficients) at point x_0, the comparison of the osculating curves should be continued with usage of the third derivative[6].

THEOREM 2. If *the curvatures of the osculating functions (5) and $f(x)$ at point x_0 are the same then the following three statements are equivalent:*

1) *At point x_0 the inequality for the third derivatives holds: $f''' > (<) g'''$;*
2) *At point x_0 the inequality which connects the Arrow-Pratt and the prudence coefficients holds:*

$$r_{f'} - r_f - 1 > (<) 0; \quad (7)$$

3) *At point x_0 the Arrow-Pratt coefficient r_f increases (decreases).*

Proof: See the Appendix.

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[5] A general sufficient condition for the inelasticity is provided by Lemma 3.

[6] In geometry one uses characteristics of curves $y = f(x)$, such as the aberrancy (a measure of non-circularity of a curve), which, given x_0, $f(x_0)$, $f'(x_0)$, $f''(x_0)$, is a monotonic function of $f'''(x_0)$ (see, e.g., Schot 1978; Gordon 2005).

4. THE USAGE OF THE CES PROTOTYPE FORM

Now let us use as prototype form the intensive form of the CES function, $g(x, A, \alpha, p) == A(\alpha x^p + \beta)^{\frac{1}{p}}$, where $\beta = 1-\alpha$, with three parameters: $A > 0, 0 < \alpha < 1,\ p \in (-\infty, 0) \cup (0,1)$. Let $f(x_0)$, $f'(x_0)$, $f''(x_0)$ be given at point x_0.

The system (4) turns into the system of equations with unknowns A, α and p:

$$\begin{cases} A(ax_0^p + \beta)^{\frac{1}{p}} = f(x_0), \\ A(\alpha x_0^p + \beta)^{\frac{1}{p}-1} ax_0^{p-1} = f'(x_0) \\ A(p-1)a\beta x_0^{p-2}(\alpha x_0^p + \beta)^{\frac{1}{p}-2} = f''(x_0) \end{cases}.$$

The solution of the first two equations of the system is a set of the CES functions with various values of p and with parameter α dependent on p [7]:

$$y = f(x_0)\left(\varepsilon_f(x_0)\left(\frac{x}{x_0}\right)^p + (1-\varepsilon_f(x_0))\right). \quad (8)$$

The third equation of the system is nothing else but the identity (2) for production functions possessing CRS (Corollary 2). Since for the CES function $\sigma = 1/(1-p)$, equation (2) implies

$$p = \frac{1-\varepsilon_f - r_f}{1-\varepsilon_f}. \quad (9)$$

The osculating function (8) with parameter p defined by (9), has at point x_0 the same values of function and its two variables (as well as the same elasticity, the curvature, and the Arrow-Price index) as function f has. Further, a comparison of the third derivatives of these functions is needed.

THEOREM 3. *If curvatures of the osculating functions f and g are equal at point* x_0, *then inequalities* $f''' > (<) g'''$ *are equivalent to*

$$r_{f'} > (<) 1 + r_f + \left(1 - \frac{r_f}{1-\varepsilon_f}\right)\varepsilon_f \quad (10)$$

at this point.

Proof: See the Appendix.

Theorem 3 provides a more subtle result than Theorem 2. However, note that in the case considered in Theorem 2 the osculating function has $r_f = 1-\varepsilon_f$; thus, inequalities (10) and (7) are consistent with each other.

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[7] Such functions as (8) are known as normalized CES functions (see Klump and de La Grandville 2000; Klump et al. 2012).

5. EXAMPLE: CLOSED ECONOMY WITH A NON-HOMOTHETIC PRODUCTION FUNCTION

As an example enlightening the role of the relations between the elasticity, the Arrow-Pratt coefficient of relative risk aversion and the coefficient of relative prudence in a non-stochastic context, let us consider a simple macroeconomic model.

Production is described by the following separable production function generalizing the Cobb-Douglas function:

$$Y = F(K,\, L) = f(K)h(L)\,. \quad (11)$$

Here Y is output, K is capital, L is labor; the functions $f(.)$ and $h(.)$ are defined and positive on some rays, $(\overline{K}, +\infty)$ and $(\overline{L}, +\infty)$; and $f'(.) > 0,\ f''(.) < 0,\ h'(.) > 0,\ h''(.) < 0$.

In a closed economy, under the assumption of perfect competition and a no-depreciation condition, all feasible bunches of production factors, $(K,\, L)$, have to satisfy the familiar balance identity

$$Y = F(K,\, L) = \frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L; \quad (12)$$

and $r = \dfrac{\partial F}{\partial K},\ w = \dfrac{\partial F}{\partial L}$ are the factor prices. If the function $F(K,\, L)$ is linearly homogeneous (CRS) then (12) follows automatically from the Euler theorem. In the general case (11), equation (12) is not automatically satisfied.

Equation (12) is equivalent to

$$\varepsilon_f(K) + \varepsilon_h(L) = 1\,, \quad (13)$$

where $\varepsilon_f(.),\ \varepsilon_h(.)$ are the elasticities of functions $f(.)$ and $h(.)$, correspondingly. Equation (13) describes the curve such that, under perfect competition, the economy is not able to operate in any points $(K,\, L)$ except the points of the curve (13)[8].

The model allows to predict structural changes in the economy. The key question of development of the economy with production function (13) is the following: will the growth of capital be accompanied by the growth of labor because of involvement of people who previously were not in labor force or were unemployed, and because of migrants (so called *extensive growth*), or, on the contrary, the growth of capital will be accompanied by disengagement of labor (*intensive growth*)?

When the curve (13) exists, this question is reduced to the following: does this curve have a positive slope (the production factors are complements) or a negative slope (the production factors are substitutes)? The answer to this question is based on relations between the elasticities and the Arrow-Pratt relative risk aversion coefficients for functions *f* and *h*.

PROPOSITION 2. *The curve (13) has positive (negative) slope at point* $(K,\, L)$ *,i.e .* $\dfrac{dL}{dK} > (<) 0$*, if and only if*

$$(1 - \varepsilon_f - r_f)(1 - \varepsilon_h - r_h) < (>) 0\,. \quad (14)$$

Proof: See the Appendix.

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[8] In some sense, it is similar to the case of Leontief technology in which the economy can operate without any losses only on a certain ray in space $(K,\, L)$.

By use of (2), the inequality (14) can be written in terms of the elasticities of substitution as

$$(\sigma_f - 1)(\sigma_h - 1) < (>) 0.$$

The other question is whether the amount of labor used in production increases or decreases when the amount of capital increases. In other words, is the curve (13) convex downwards or convex upwards?

PROPOSITION 3. *The curve (13) is convex downwards (convex upwards) at point* (K, L) *if and only if the following inequalities are fulfilled*[9]*:*

$$2\varepsilon_f + 2r_f - 2\varepsilon_f^2 - 3\varepsilon_f r_f - r_f r_{f'} \le (\ge) 0,$$
$$2\varepsilon_h + 2r_h - 2\varepsilon_h^2 - 3\varepsilon_h r_h - r_h r_{h'} \le (\ge) 0,$$

Proof: See the Appendix.

As an example, let us consider the production function of type (11):

$$Y = F(K, L) = \ln K \ln L,$$

where $K > 1,\ L > 1$. It is easy to calculate $\varepsilon_f = 1/\ln K,\ \varepsilon_h = 1/\ln L,\ r_f = r_h = 1,\ r_{f'} = r_{h'} = 2.$ The inequality (14) takes the form $1/(\ln K \ln L) > 0;$ and the inequalities in Proposition 3 are, correspondingly, $-1/\ln K - 2/(\ln K)^2 < 0$ and $-1/\ln L - 2/(\ln L)^2 < 0$. These inequalities imply that the curve (13) has negative slope and is strictly convex downwards. Correspondingly, increase in capital is accompanied in a long run by decrease in employment. If $K \to +\infty,$ then, by (13), $L \to e.$ Hence, if the labor force in this economy is L_0 then the long-run unemployment rate is $(1 - e/L_0).$

6. CONCLUSION

The concept of elasticity of substitution plays an important place in economic theory.

Many economists find useful to link this concept with the geometric concept of curvature, despite no formal relation was found. De La Grandville (1998) argues that “there is no link between curvature and the elasticity of substitution”. A new trend, which appeared in economic research recently, is the active usage of the formal constructions of the Arrow-Pratt relative risk aversion coefficient and the coefficient of relative prudence as characteristics of utility functions and production functions even in non-stochastic framework. These coefficients are also claimed to have relation to curvature.

In the present paper we clarify links between the concept of curvature and concepts of the elasticity of substitution, the elasticity, the Arrow-Pratt relative risk aversion coefficient and the coefficient of relative prudence. We show that these notions relate to the same geometric construction. To explain it we suggest a simple approach based on notions of prototype forms and osculating curves.

Different types of functions/curves can be used as prototype forms. If the prototype form corresponds to a circle, we come to the classic geometric concept of curvature. If the prototype form is the power function (what is the same as the intensive form of the Cobb-Douglas function or the CRRA function) or the CES function then we come to expressions containing the Arrow-Pratt coefficient of relative risk aversion and the coefficient of relative prudence. The CES function in such case, naturally provides also a value of elasticity of substitution as an important characteristic closely connected with other characteristics of the production function or utility function. Equation

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9 Similar pairs of strict inequalities provide necessary and sufficient conditions of strict convexity downwards (upwards).

(2) links the elasticity of substitution, the elasticity, and the Arrow-Pratt coefficient of relative risk aversion.

Thus, the geometric curvature and the elasticity of substitution relate to different types to of prototype functions. Moreover, the curvature is calculated by use of the extensive form of function, while other concepts under consideration can be defined by use of either intensive or extensive forms.

We provide three theorems which describe precise relations between these concepts. From the works by Mrázová and Neary (2013) and Levine (2012) it is clear that such kind of relations can play an important role in economic analysis. To enlighten further this role we provide an example of simple macroeconomic model based on a separable production function which generalizes the Cobb-Douglas function.

LITERATURE

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APPENDIX

Proof of Lemma 1. Applying to $f'x/f$ the rules for calculating the elasticity, we immediately obtain

$$\varepsilon_{\varepsilon f} = \varepsilon_{f'} + 1 - \varepsilon_f = -r_f + 1 - \varepsilon_f .$$

By applying the same rules to $-f''x/f'$, we immediately obtain

$$\varepsilon_{rf} = -\varepsilon_{f''} - 1 + \varepsilon_{f'} = r_{f'} - 1 - r_f .$$

Q.E.D.

Proof of Lemma 2. Differentiating $F(X_1, X_2) = X_2^{\gamma} f(x),$ we find

$$\frac{\partial F}{\partial X_1} = X_2^{\gamma-1} f'(x),\ \frac{\partial F}{\partial X_2} = \gamma X_2^{\gamma-1} f(x) - X_2^{\gamma-2} f'(x) X_1;$$

it follows that,

$$S_{12} = \frac{\gamma f(x) - f'(x)\cdot x}{f'(x)} = x\left(\frac{\gamma}{\varepsilon_f} - 1\right).$$

Q.E.D.

Proof of Proposition 1. Let us consider the inverse, $1/\sigma$, i.e. the elasticity of S_{12} with respect to *x*. Using the rules for calculating the elasticities, according to the formula (1), we find this elasticity as a combination of elasticities of functions $x,\ (\gamma - \varepsilon_f)$ and ε_f with respect to x:

$$\frac{1}{\sigma} = 1 - \frac{\varepsilon_f'}{\gamma - \varepsilon_f} x - \frac{\varepsilon_f'}{\varepsilon_f} x =$$

$$= 1 - \frac{\varepsilon_f' x\gamma}{(\gamma - \varepsilon_f)\varepsilon_f} = 1 - \frac{\gamma}{\gamma - \varepsilon_f} \varepsilon_{\varepsilon f} .$$

Using Lemma 1, we get

$$\sigma = \frac{1}{1 - \dfrac{\gamma}{\gamma - \varepsilon_f}(-r_f + 1 - \varepsilon_f)} = \frac{\gamma - \varepsilon_f}{(1-\gamma)\varepsilon_f + \gamma r_f}.$$

Q.E.D.

Proof of Lemma 3. The function $f(.)$ is differentiable and strictly convex, hence (see Takayama 1994, p. 57), $f'(x_0)(x - x_0) > f(x) - f(x_0)$ for all $x, x_0 \in (0,+\infty),\ x \neq x_0$. If $x < x_0$ then

$$f'(x_0) < \frac{f(x) - f(x_0)}{x - x_0}.$$

The R.H.S. decreases in x and converges to $f'(x_0)$ as $x \to x_0$. Hence, as $x \to 0$, the R.H.S is strictly greater than $f'(x_0)$; thus, $f'(x_0) < f(x_0)/x_0$ for each $x_0 \in (0,+\infty)$. Q.E.D.

Proof of Theorem 1. Since $f'(x_0) = g'(x_0)$, to compare the curvatures it is sufficient to compare the second derivatives $g''(x_0)$ and $f''(x_0)$. By assumption (3), $f''(x_0) < 0$. The sign of g'' is determined by $(e_f - 1)$ which, by assumption, is negative, i.e. $g'' < 0$. Hence, function (5) has higher (lower) curvature than function $f(x)$ if and only if

$$f'' > (<) g''. \quad (15)$$

But

$$g' = \frac{f(x_0)\varepsilon_f(x_0)}{x_0^{e_f(x_0)}} x^{e_f(x_0)-1}, \quad g'' = \frac{f(x_0)\varepsilon_f(x_0)(\varepsilon_f(x_0)-1)}{x_0^{e_f(x_0)}} {x_0}^{e_f(x_0)-2},$$

Hence, at point x_0, $g'' = f'(x_0)(\varepsilon_f(x_0)-1)/x_0$, and inequalities (15) are equivalent to the following:

$$\frac{f''(x_0)x_0}{f'(x_0)} > (<) \varepsilon_f - 1$$

which is the same as (6). Thus, the statements 1) and 3) are equivalent.

Lemma 1 implies the equivalence of the statements 2) and 3). Q.E.D.

Proof of Theorem 2. Equality of curvatures means that

$$-r_f(x_0) - \varepsilon_f(x_0) + 1 = 0, \quad (16)$$

Taking (16) into account, we obtain

$$g'''(x_0) = \frac{f(x_0)\varepsilon_f(x_0)(\varepsilon_f(x_0)-1)(\varepsilon_f(x_0)-2)}{x_0^{e_f(x_0)}} x^{\varepsilon_f(x_0)-3} = \frac{f''(x_0)}{x_0}(\varepsilon_f - 2).$$

Hence, accounting for the sign of the second derivative, inequalities $f''' > (<) g'''$ are equivalent to $f'''(x_0)x_0 / f''(x_0) < (>) \varepsilon_f - 2$, which, recalling (16), is equivalent to (7); thus, the statements 1) and 2) are equivalent.

Lemma 1 implies the equivalence of the statements 2) and 3).
Q.E.D.

Proof of Theorem 3. The third derivative is

$$g'''(x_0) = (p-1)ab[(1-2b)({ax_0}^p + b)^{\frac{1}{p}-3} a{x_0}^{2p-3} + ({ax_0}^p + b)^{\frac{1}{p-2}}(p-2){x_0}^{p-3}] =$$

$$= \frac{g''(x_0)}{x_0}\left(-1 - \frac{p{ax_0}^p}{{ax_0}^p + b} + \frac{(p-1)b}{{ax_0}^p + b}\right) = \frac{g''(x_0)}{x_0}\left[-1 - \left(1 - \frac{r_f(x_0)}{1-\varepsilon_f(x_0)}\right)\varepsilon_f(x_0) - r_f(x_0)\right].$$

Thus, accounting for the sign of the second derivative, the inequalities $f''' > (<) g'''$ are equivalent to

$$\frac{f'''(x_0)x_0}{f''(x_0)} < (>) -1 - r_f - \left(1 - \frac{r_f(x_0)}{1-\varepsilon_f(x_0)}\right)\varepsilon_f(x_0)$$

and, well then, (10). Q.E.D.

Proof of Proposition 2. Let us consider the L.H.S. of equation (13) as a function of two variables and apply the implicit function theorem: $dL/dK = -\varepsilon_f'(K)/\varepsilon_h'(L)$. The inequalities $dL/dK > (<)0$ are equivalent to

$$\varepsilon_f'(K)\varepsilon_h'(L) < (>)0 \text{ ; } (17)$$

Corollary 1 implies that (17) is equivalent to (14). Q.E.D.

Proof of Proposition 3. For the function in the L.H.S. of (13), $\varepsilon_f(K) + \varepsilon_h(L)$, the Hessian matrix is

$$\mathbf{H} = \begin{pmatrix} \varepsilon_f''(K) & 0 \\ 0 & \varepsilon_h''(L) \end{pmatrix}.$$

This matrix is positive (negative) semidefinite and, correspondingly, the curve (13) is convex downwards (convex upwards) if $\varepsilon_f''(K) \geq (\leq)0,\ \varepsilon_h''(L) \geq (\leq)0$.

By use of the expression for $\varepsilon_{f'}$ obtained in Corollary 1, we calculate

$$\varepsilon_f''(K) = \frac{(1 - r_f(K) - \varepsilon_f(K))(f''(K)f(K) - 2(f'(K))^2) + (1 - r_{f'}(K) + r_f(K))f'f}{f^2},$$

which has the same sign as

$$-\left[(1 - r_f(K) - \varepsilon_f(K))\left(1 + 2\frac{\varepsilon_f(K)}{r_f(L)}\right) + (1 - r_{f'}(K) + r_f(K))\right].$$

Similarly, for $\varepsilon_h''(L)$. This leads to the assertion of the Lemma. Q.E.D.