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EXTREMAL QUANTILES OF MAXIMUMS FOR STATIONARY SEQUENCES WITH PSEUDO-STATIONARY TREND WITH APPLICATIONS IN ELECTRICITY CONSUMPTION

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ABSTRACT

We propose a method for estimation of the distribution function of the maximums for time series with pseudo-stationarytrend on the basis of the earlier proved by the author theorems. The results are applied for estimation of the distribution function for the extremal values of electrical energy consumption.

KEY WORDS: *Stationary sequences with periodic trend, estimation of the distribution function, forecasting, energy consumption.*

JEL Classification: C61, C62, C63, C72
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1. INTRODUCTION

The problem of estimation the distribution function for the maxima of time series with pseudo-trend plays an important role when calculating reserves, forecasting consumption peaks (e.g., energy consumption), predicting extremes in weather events (e.g., temperatures), forecasting extremal price-levels. These challenges can be approached from the perspective of the classical results from Extreme Values Theory (EVT) (see Suveges 2008), and from a position of the results which, to some extent, are extensions of the classical EVT, seasonally adjusted data. We study the behaviour of maximums of electrical energy consumption in Russia from the point of both methods. But it should be stresses that the second extanded method allow us to use more data and get robust results. For empirical illustrations we use hourly electricity consumptionin Russia from the period from 1-th of July till 10-th of September 2005, taken from the site of the System Operator of the Unified Energy System of Russia (see 1).

The rest of the paper proceeds as follows. Section 1 provides formulations of the theorems for approximation the distribution function of maximum for time series with pseudo-stationary trend. In theoretical background section, we presented Fisher, Tippett and Gnedenko theorem together with the extended limit theorem for normalized maxima for stationary sequences with pseudo-stationary trend. Section 2 consists of empirical illustrations where is presented the procedures for high quantiles approximation. Section 3 concludes.

2. THEORETICAL BACKGROUND

The classical Extreme Value Theory studies asymptotic distribution of maxima of independent and identically distributed random variables with the function with distribution function $F(x)$.The basis of this theory is Fisher-Tippett-Gnedenkotheorem (Theorem on extremal types, see (De Haan and Ferreira 2006), (Fisher and Tippet 1928), (Gnedenko 1943), (Leadbetter and Lingren et al. 1983). Theorem Fisher-Tippet-Gnedenko, states:

Theorem 1.n (Gnedenkon 1943); Fisher and Tippet 1928)
If for the distribution functions $F(x)$ and $H(x)$ there are such $a_n > 0$ and b_n, that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = H(x),$$

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at any continuity point of $H(x)$, then $H(x)$ must coincide up to linear transformation of the argument x (with a positive coefficient of scale) with one of the three distribution functions, $H_1(x) := exp\{-e^{-x}\}$ (Gumbel distribution), $H_2(x) := exp\{-x^{\beta}\}, x > 0$, (Frechet distribution with $\beta < 0$) and $H_3(x) := exp\{-(-x)^{\beta}\}$, $x \leq 0$ (Weibull distribution with $\beta > 0$.) .

Note, that $F^n(a_n x + b_n)$ represents the distribution function of the normalized maximum for n identically independently random variables with distribution function F.
Denote D_ν, $\nu = 1,2,3$, as the disjoint domains of attraction, that is $F \in D_\nu$ if and only if the limit of the sequence $F^n(a_n x + b_n)$ belongs to type H_ν, $\nu = 1,2,3$, respectively. Details and proofs can be found in the monograph (Leadbetter M. R. et al. 1983). It turns out, that this approximation is suitable in the case of weak dependance for the large values of the X_i far away from each other, see (Leadbetter et al. 1983, Leadbetter 1974) Relevant mixing conditions are known as Leadbetter's mixing conditions.

In this paper we consider the extanded problem of approximation for distribution of the maximum values of the time series:

$$Y_i = X_i + cm_i, \quad i = 1,2,\ldots, \tag{1}$$

where $\{X_i,\ i = 1,2,\ldots\}$ – strictly stationary random sequence, $\{m_i,\ i = 1,2,\ldots\}$ – trend, which behaves in the below defined stationary manner (for example, seasonal component), иc – small parameter. It is assumed, that the distribution function $F(x)$ of the random variable X_1 belongs to the maximum-domain of attraction. It means, that there are some positive sequence a_n, some b_n and nondegenerate distribution function $H(x)$ such that at any continuity point x of the function H the sequence $F^n(a_n x + b_n)$ converges to $H(x)$ when $n \to \infty$.

We introduce some initial conditions which are needed for our main result.

Condition 1. *The sequence $\{m_i i = 1,2,\ldots\}$ is above-bounded: $m := sup_{i=1,2,\ldots} m_i < \infty$.*

Onwards the small parameter c is taken equal to a_n, where a_n – normalizing sequence from Fisher-Tippet-Gnedenko theorem, which corresponds to the distribution function F, that is, $c = c(n) \equiv a_n$. In the next section, we recall a specific type of normalization (a_n, b_n), depending on the attraction domains, D_1,D_2 or D_3 for distribution function F. More details see, for example, in (7).

Denote $u_n = a_n x + b_n$.Introduce Leadbetter's type mixing condition for large values of the (1).

Condition 2. *(Condition $D^2(u_n, a_n, \{m_k\}_{k=1,\ldots,n})$) There exists a family of numbers $\{\alpha_{n,l}\}$, $n, l = 1,2,\ldots$ and sequence of positive integer numbers $\{l_n\}$ such that $l_n = o(n)$, $\alpha_{n,l_n} \to 0$, and for any x, y and arbitrary sets of positive integer numbers $I = \{i_1,\ldots,i_p\}$,$J = \{j_1,\ldots,j_q\}$ such that*

$$1 \leq i_1 < i_2 <\ldots< i_p < j_1 <\ldots< j_q \leq n, \quad j_1 - i_p \geq l_n,$$

holds the following inequality:

$$|P(\cap_{j\in I\cup J} \{X_j \leq u_n - a_n m_j\}) -$$
$$-P(\cap_{j\in I} \{X_j \leq u_n - a_n m_j\})P(\cap_{j\in J} \{X_j \leq u_n - a_n m_j\})| \leq \alpha_{l_n,n}.$$

Condition 2 ensures the mixing (weak dependence) far away separated large values of the time series (1).

Condition 3. *(Condition $D'(u_n - ma_n)$) The following equality holds:*

$$\lim_{k\to\infty}\limsup_{n\to\infty} n \sum_{2\le j\le n/k} P\{X_1 > u_n - ma_n; X_j > u_n - ma_n\} = 0.$$

We introduce " empirical distribution functions" of the trend values for Y_i:

$$G_n(x) = \frac{\#\{i: m_i \le x, 1 \le i \le n\}}{n},$$

where the sign # denotes the number of elements of the set.

Let G – nondecreasing nonnegative left-continuous bounded function, denote $a_+ := \max(a, 0)$, and define the functions:

$$L_1(z,G) = e^{-z}\int_{-\infty}^{+\infty} e^t dG(t);$$
$$L_2(z,G) = \int_{-\infty}^{+\infty} (z-t)_+^{\beta} dG(t),\ \beta < 0;$$
$$L_3(z,G) = \int_{-\infty}^{+\infty} (t-z)_+^{\beta} dG(t),\ \beta > 0.$$

Now we formulate the pseudostationarity condition for the sequence $\{m_k,\ k = 1,2,\ldots\}$.

Condition 4. There exists $\mathrm{G(x)}$ such that the convergence in probability holds:

$$\lim_{n\to\infty} G_n(x) = G(x) \qquad (2)$$

at any continuity points x of the function $\mathrm{G(x)}$. Besides, for any $\nu = 1,2,3$, if $\mathrm{F} \in \mathrm{D}_\nu$, then for any x and $\eta = 0,1$ there exists finite limits:

$$\lim_{n\to\infty} L_\nu(x, G_n) = L_\nu(x, G) < \infty.$$

Functions $L_\nu(x,G)$ are involved in formulas for limit distribution of the maxima. Note, that if (2) is satisfied, then $L_2(x,G)$ isn't necessarily finite.

The main result of the article concerns the limiting joint distribution of the random variables:

$$M_n = \max\{X_i + m_i a_n;\ i = 1,\ldots,n\}$$

with infinitely growing n.

Theorem 2. *Let in model (1) $F \in D_\nu$, where $\nu = 1,\ 2$ or 3. Assume, that conditions $1-4$ are satisfied. Then, if $\nu = 1$ or $\nu = 3$, then for any x, y,*

$$\lim_{n\to\infty} P\{M_n \le u_n\} = e^{-L_\nu(x,G)}, \qquad (3)$$

if $\nu = 2$, then for any $x, y > m$,

$$\lim_{n\to\infty} P(M_n \le u_n) = e^{-L_2(x,G)}. \qquad (4)$$

Proof of Theorem 2: Proof of this theorem is similar to (Kudrov 2008).

3. EXTREMAL BEHAVIOUR OF ELECTRICAL ENERGY CONSUMPTION

In this section we apply our theoretical results for the study of hourly consumption of electrical energy in Russia for the period from 1-th of July till 10-th of September 2005 year. Visual analysis of changes in electrical energy consumption leads to conclusion about its periodicity per day.

Moreover, we can see that there is a periodicity associated with days within a week, and yearly-periodicity (seasonal homogeneity), so that it is common, that changing in electrical energy consumption during the year follows the seasonal regularity, and, for example, we can distinguish the months with the highest electrical energy consumption and the months with the lowest electrical energy consumption. For the full study of extremal electrical energy consumption we must also consider yearly-trend.

We take the data from subperiod, 7-th of June till 22-th of July 2005, as an example of seasonal homogeneity. We will consider only the data from Tuesday to Thursday in every week since consumption peaks during a week is reached only in these days and for these days there is a similar structure of consumption.

We denote by $\hat{C}_k$ consumption for the k-th hour of the considered time-interval. Let $(\hat{C}_k)$ is a sample from stochastic sequence (C_k). Assume that the elements of this random sequence (C_k) is represented as the sum of deterministic periodic component (p_k) and a stationary time series (X_k) with zero mean, otherwise it can be subtracted from stationary stochastic component and added to the deterministic component:

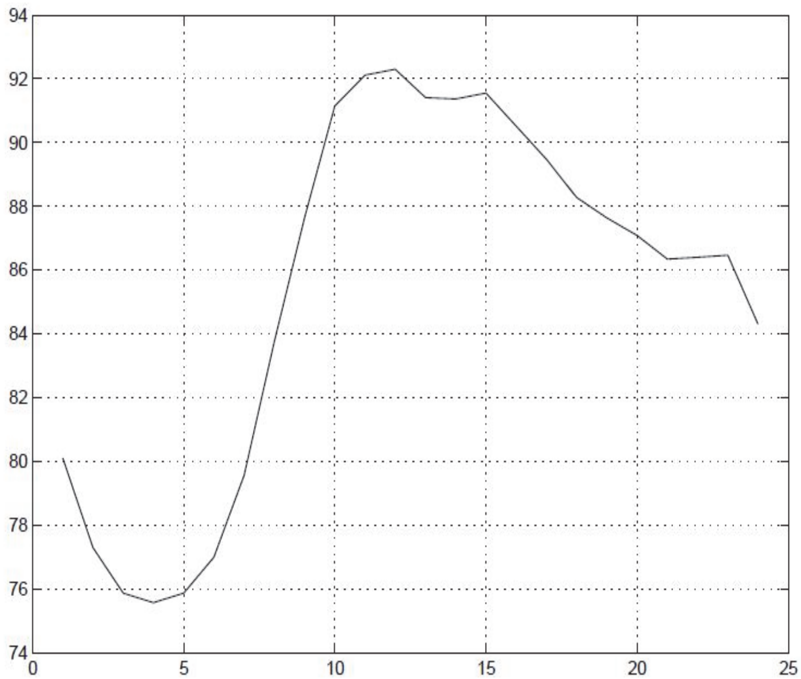
$$C_k = X_k + p_k.$$

Next, we suppose that a deterministic periodic component has a period, which equals to 24 which corresponds to the number of hours per day. The estimator for (p_k) is calculated as follows:

$$\hat{p}_i = \frac{\hat{C}_i + \hat{C}_{i+24} + \ldots + \hat{C}_{i+24(K-1)}}{K}, \qquad (12)$$

where $1 \le i \le 24, i \in N$ and K – number of days covered by the sample (in this case, K = 28). Figure 1 shows graphically the values $(p_i : 1 \le i \le 24)$.

Figure 1: *Estimator for periodic component (energy consumption)*



Denote

$$\hat{X}_i = \hat{C}_i - \hat{p}_i, 1 \le i \le 24K.$$

Since consumption peaks during a day occur in the time interval between $8:00$ and $18:00$, it makes sense to consider only the values which correspond to this period of time, namely:

$$(\hat{C}_{i+24(m-1)}, i \in [8,18] \cap \mathbb{N}, m = 1, \dots, K),$$

$$(\hat{X}_{i+24(m-1)}, i \in [8,18] \cap \mathbb{N}, m = 1, \dots, K),$$

$$(\hat{p}_i, i \in [8,18] \cap \mathbb{N}).$$

Denote j-th element of the first two sequences from the above mentioned as $\hat{C}_j^*$, $\hat{X}_j^*$, respectively, where $1 \le j \le 11K$, and j-th element of the third sequence as $\hat{p}_j^*$, where $1 \le j \le 11$.

We take the maximal element in each interval of indices $[1 + 11(m-1), 11m]$, where $m = 1, .., K$ for the sequences $(\hat{C}_j^*)$ and $(\hat{X}_j^*)$, which we denote as:

$$\widehat{M}_1, \dots, \widehat{M}_K$$

and

$$\widehat{M}_1', \dots, \widehat{M}_K',$$

respectively.

Let $(\hat{X}_j^*)$ - sample from (X_j^*), then $(\hat{C}_j^*)$ - sample from $(X_j^* + \hat{p}_j^*)$. Assume, that (X_j^*) is stationary and satisfies asymptotic independence property. Then, applying the results of Theorem 4 (the case, when periodical component equals zero), we get the limit distribution function for the lineraly normalized maximum of (X_j^*) (distribution function of extremal types). We estimate the parameters of this limit theoretical distributional function. For that we need to estimate an extremal index for the distribution function of extremal types. We use Pickands estimator for the extremal index.

Let

$$\hat{X}_{11K,11K}^* \le \hat{X}_{11K-1,11K}^* \le \dots \le \hat{X}_{1,11K}^*,$$

– order statistics for $(\hat{X}_j^*)$.

Then Pickand's estimator is defined as follows:

$$\hat{\xi}_{i,n} = \frac{1}{\ln 2} \ln \frac{\hat{X}_{i,11K}^* - \hat{X}_{2i,11K}^*}{\hat{X}_{2i,11K}^* - \hat{X}_{4i,11K}^*}$$

This estimator has the following properties (see DeHaan 2005):

1) If $i(n)/n \to 0$ when $n \to \infty$, then $\hat{\xi}_{i,n}$ converges in probability to ξ (consistent estimator).

2) Under some additional conditions $\sqrt{i}(\hat{\xi}_{i,n} - \xi)$ has asymptotically normal distribution with zero mean and variance:

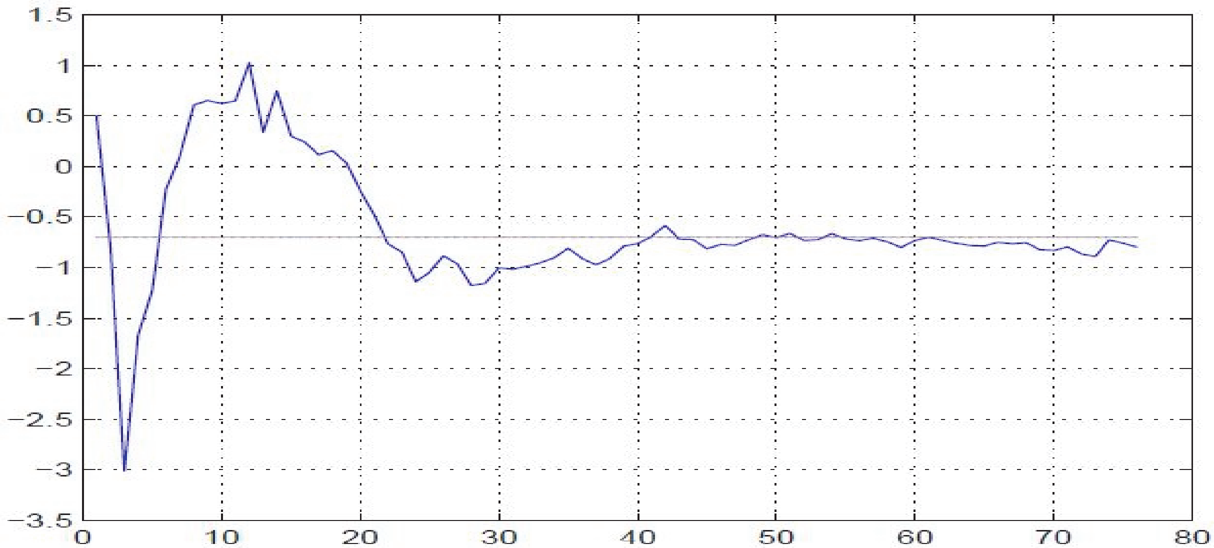
$$v(\xi) = \frac{\xi^2(2^{2\xi+1}+1)}{(2(2^{\xi}-1)\ln 2)^2}.$$

For the choice of optimal Pickand's estimator $\hat{\xi}_{i,n}$, we use visual method, for that we depict the graph for

$$\{(i, \hat{\xi}_{i,n}): i = 1, \ldots, 101K/4\},$$

see Figure 2.

Figure 2: The graph for Pickands estimator for data with subtracted periodic component



and choosing the largest area where the graph is nearly horizontal (see [Embrechts, Kluppelberg, Mikosch (1999)]). Thus, we take:

$$\tilde{\xi} = -0.7009,$$

in accordance with the above-mentioned properties, 95% asymptotic confidence interval for this value is:

$$[-1.5776; -0.5967].$$

Denote the empirical distribution function $\widehat{M}_1, \ldots, \widehat{M}_K$ as $U(x)$, and the empirical function distribution of $\widehat{M}_1^{'}, \ldots, \widehat{M}_K^{'}$ as $G(x)$.

Let us now compare the empirical distribution function $G(x)$ and the theoretical distribution function with extreme type index $\tilde{\xi}$. For this we use the QQ −plot to depict graphically the set (see Fig. 3):

$$A = \left\{ \left(G^{-1}(i/(K+1)); -\left(-\ln\left(\frac{i}{K+1}\right)^{-\tilde{\xi}} \right) \right): i = 0, \ldots, K \right\}.$$

As we can see the elements of set A are very close to the line constructed using the method of weighted least squares. Transform linearly y-axis on the coordinate plane replacing it by $(y^{'} = (y - b)/a)$ so that the points set A are located along the line $y^{'} = x$ (see Fig. 4).

Figure 3: Quantile-quantile plot for A, where on x-axis are pointed the quantiles of empirical distribution function for normalized maximums of data with subtracted periodic trend and on y-axis are pointed the quantiles of standard distribution function of extreme types with the estimated extremal index $\tilde{\xi}$

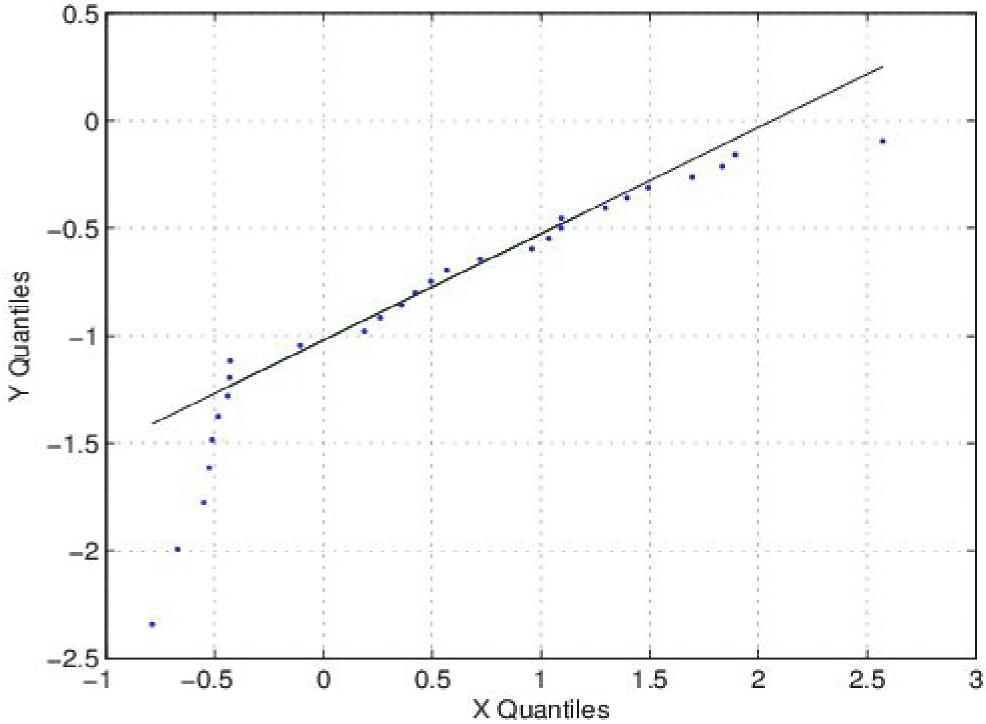
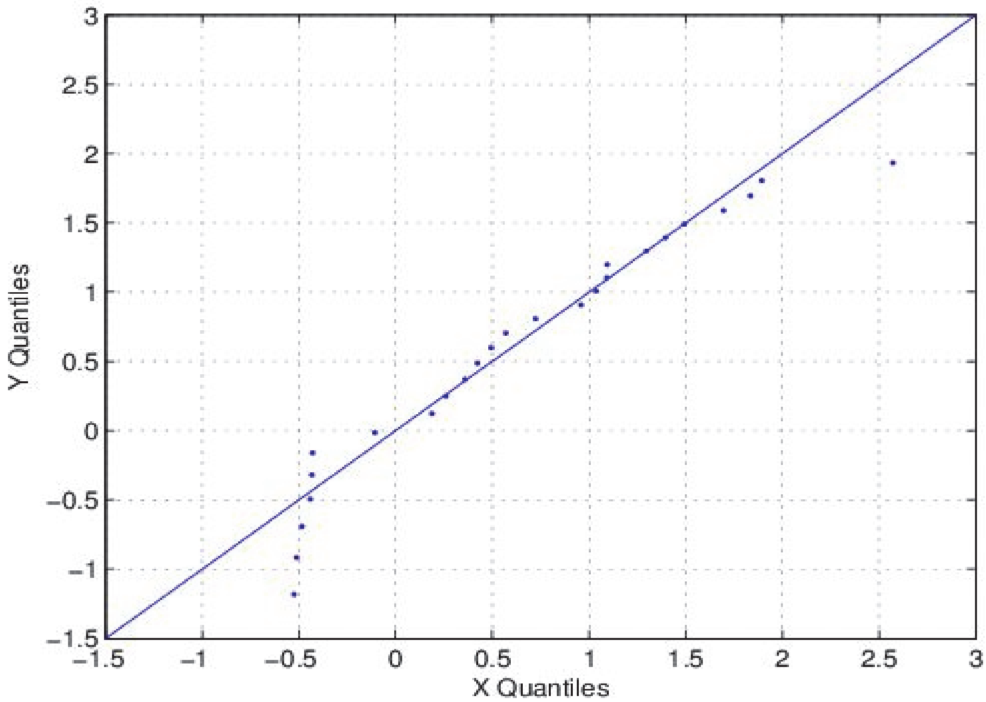


Figure 4: The graph obtained after a linear transformation of the second coordinate for elements of A, where on x-axis are pointed the quantiles of empirical distribution function for normalized maximums of data with subtracted periodic trend, and on y-axis are pointed the linearly transformed quantiles of standard distribution function of extremal types with extremal index $\tilde{\xi}$



This linear transformation defines normalization for maximums by which we normalize maximums:

$$(\widehat{M}_1 - b)/a, \ldots, (\widehat{M}_K - b)/a.$$

Next, we use Theorem 2, wherein is presented limiting distribution function for normalized maxima for corresponding sequences (including periodic component) and get that the distribution function

$$P(x) = \exp\left\{-\frac{1}{11}\sum_{i:\frac{\hat{p}_i^*}{b}>x}\left(\frac{\hat{p}_i^*}{b} - x\right)^{-\tilde{\xi}}\right\}$$

should approximate the empirical distribution function of the sample:

$$(\widehat{M}_1 - b)/a, \ldots, (\widehat{M}_K - b)/a.$$

In order to see how well one distribution function approximates another distribution function, refer to the set:

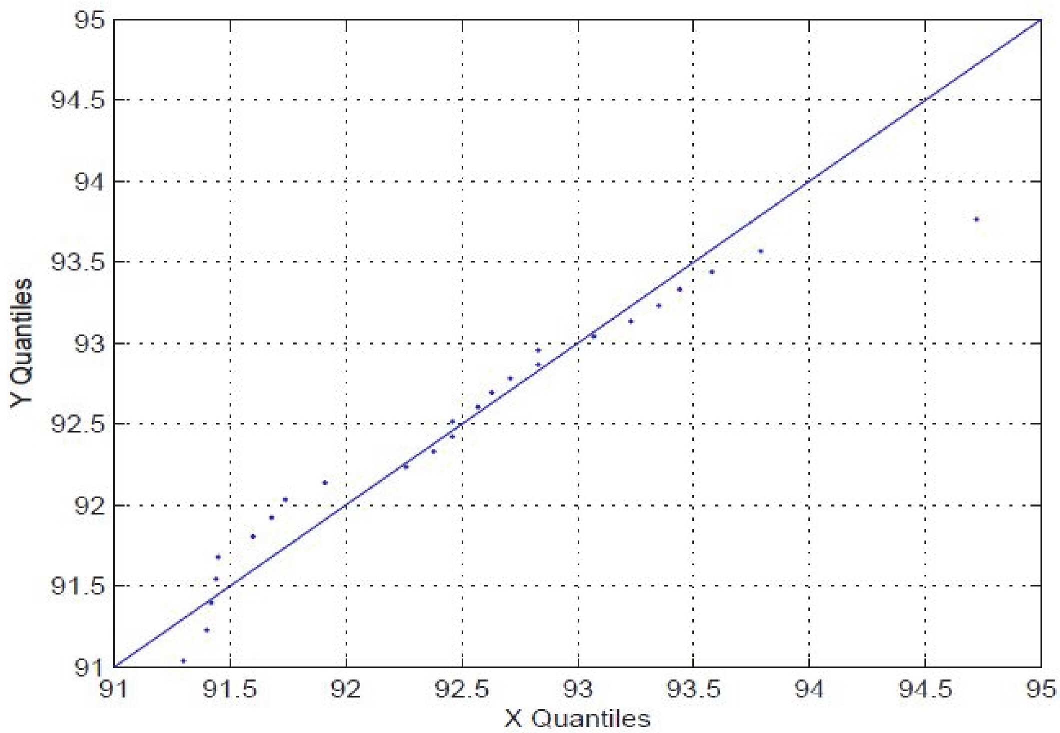
$$B = \{(U^{-1}(i/(K+1)); at(i/(K+1)) + b): i = 0, \ldots, K\},$$

where $t(i/(K+1))$ – solution of the equation:

$$\exp\left\{-\frac{1}{11}\sum_{i:\frac{\hat{p}_i^*}{b}>t(i/(K+1))}\left(\frac{\hat{p}_i^*}{b} - t(i/(K+1))\right)^{-\tilde{\xi}}\right\} = \frac{i}{K+1}.$$

Note that this equation always has a solution, as a function on the left-hand side is monotone in $t(i/(K+1))$. Let point on (x, y)-plane the graph for B (see Fig. 5).

Figure 5: Quantile-quantile plot for B, where on x-axis are pointed the quantiles of empirical distribution function of the normalized maximums and on y-axis are pointed the quantiles of theoretical distribution function from Theorem 2, taking into account the periodic component



As we can see from the figure, the points of B are located sufficiently close to the line $y = x$, which means that the distribution of $P(ax + b)$ quite accurately approximates the empirical distribution function $U(x)$in the region of high quntiles.

Since the periodic component in the considered time interval (from 8:00 to 18:00) is sufficiently flat, it seems reasonable to consider the application of the classical extreme value theory, excluding the impact of the trend, and compare that results with the results obtained above.

In order to construct an estimate of extreme index we use Pickands estimator again using observations:

$$\hat{C}_1^*, \dots, \hat{C}_{11K}^*.$$

Take an order statistics for the sequence $(\widehat{M}_i)_{i=1}^K$:

$$\hat{C}_{11K,11K}^* \le \hat{C}_{11K-1,11K}^* \le \dots \le \hat{C}_{1,11K}^*,$$

then Pickands estimator for extremal index is:

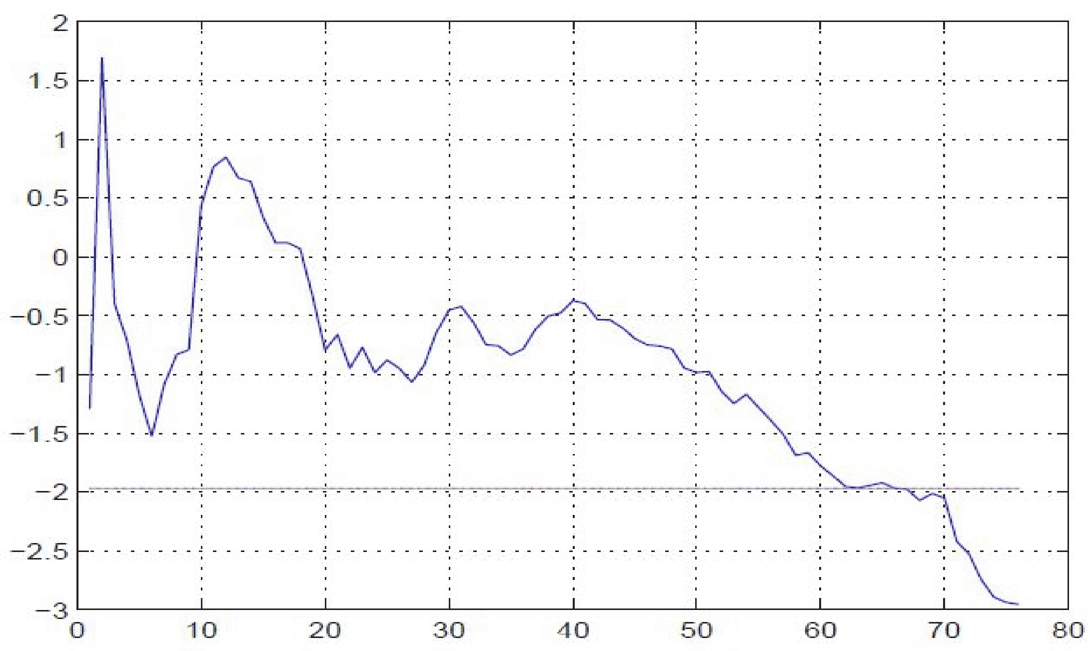
$$\hat{\eta}_{i,n} = \frac{1}{\ln 2} \ln \frac{\hat{C}_{i,11K}^* - \hat{C}_{2i,11K}^*}{\hat{C}_{2i,11K}^* - \hat{C}_{4i,11K}^*},$$

where $1 \le i \le 11K/4$.

On Figure 6 it is shown a graph of the set:

$$\{(i, \hat{\eta}_{i,n}) : i \in \overline{1; 11K/4}\}.$$

Figure 6: The graph for Pickands estimator for daily maximums



In accordance with the above mentioned procedure, we select Pickands estimator for extremal index:

$$\tilde{\eta} = -1.9690,$$

And 95% asymptotic confidence interval for that value:

$$[-2.2592; -1.6788].$$

Note that this estimate of extremal index differs significantly from the extremal index which we got for the data with subtracted periodic component.

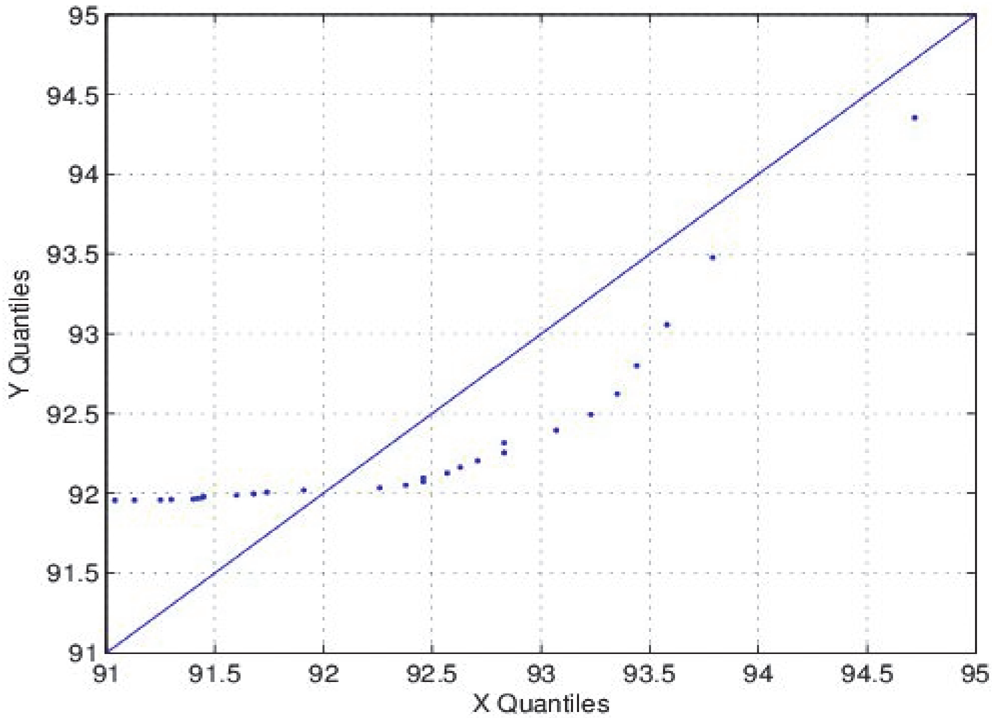
Define a linear normalization (y = ax + b) using method ofweighted least squares for the following 2-dimensional data:

$$C = \left\{\left(-\left(-\ln\left(\frac{i}{K+1}\right)^{-\tilde{\eta}}\right); G^{-1}(i/(K+1))\right) : i = \overline{0, K}\right\}$$

On Figure 7 we point the graph for the set:

$$D = \left\{\left(G^{-1}(i/(K+1)); -a\left(-\ln\left(\frac{i}{K+1}\right)^{-\tilde{\eta}}\right) + b\right) : i = \overline{0, K}\right\}.$$

Figure 7: Quantile-quantile plot for D, where on x-axis are pointed the quantiles of empirical distribution function of the normalized maximums and on y-axis are pointed the quantiles of theoretical distribution function from Theorem 2, withou taking into account the periodic component on the basis of daily maximums



Comparing graphs for the sets B and D, we conclude that the inclusion of a periodic trend provides a better estimators for description the empirical distribution function of maximumsas compared with estimates constructed on the basis of a sample of daily maxima.

4. CONCLUSIONS

To the problem of estimating the distribution function for maxima of stationary sequence with periodic trend can be approached using two approaches . The first approach is based on the results of classical extreme value theory, the second approach is based on the result proved by the author which concerns limit theorem for normalized maximums of stationary sequence with periodic trend.

Significant limitation of the first approach is small number of data (maximums) to be processed. The second approach allows us to overcome this limitation, because it takes into account the presence of a periodic trend. The second approach allows us to consider more data, and therefore using it possible to obtain a more robust estimates.

For data with a periodic trend accounting for periodic component allows to obtain more accurate estimates for distribution function of maxima. In this paper it is shown as an example electricity consumption in Russia .

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