Montenegrin Journal of Economics

Vol. 13, No. 1 (2017), 141-158

Sequential Model of Economic System and Constrained Pareto Optimality

ZORAN POPOVIĆ[1], and MARKO BACKOVIĆ[2]

[1] Faculty of Economics, University of Belgrade, Serbia, e-mail: zpop@ekof.bg.ac.rs
[2] Faculty of Economics, University of Belgrade, Serbia, e-mail: backovic@ekof.bg.ac.rs

ARTICLE INFO

Received October 11, 2016
Received in revised from December 02, 2016
Accepted February 14, 2017
Available online 15 March 2017

JEL classification:

D50, D52, D61

DOI: 10.14254/1800-5845/2017.13-1.10

Keywords:

general equilibrium model,
incomplete asset markets,
existence of solution,
pareto optimality.

ABSTRACT

The concept of general economic equilibrium of sequential structures of economic system involves development of the general economic equilibrium model that includes the structure of asset markets. The assets are instruments of income redistribution through the different states of nature of economic system. General economic equilibrium model with the asset markets should provide, on the one hand, an explanation of the relative prices of commodities and, on the other hand, the possibility of analyzing income transfers through the different states of nature and the process of asset prices formation. After defining the appropriate general economic equilibrium model which involves asset markets in the absence of arbitrage process, further analysis is focused on the examination of Pareto optimality of economic system. The existence of imperfect market conditions in the form of incomplete asset markets reduces possibility of income transfers through the different states of nature of economic system, which results that equilibrium allocation is not Pareto optimal allocation. Therefore, in this paper focus is on a constrained form of Pareto optimality in a sequential model of economic system.

1. SEQUENTIAL MODEL OF ECONOMIC SYSTEM WITH ASSET MARKETS UNDER UNCERTAINTY

Researches in the field of general economic equilibrium attempt to include time dimension and a certain degree of uncertainty in the process of market exchange in already defined mathematical relations in the standard general economic equilibrium model, with the main objective of more precise description of the market processes in one economic system. General economic equilibrium models that include time dimension and a certain degree of uncertainty in the process of market exchange are based on the concept of general equilibrium of sequential structure of economic system[1].

The sequential models should in a more realistic way describe economic reality, since in these models economic system is viewed through both time and space dimensions as the possible states of the economic system. Model defined in this paper is a model of sequential market structure consisting of spot commodity markets and asset markets, where assets provide consumers (agents) ability to redistribute income through different periods of time. Unlike a perfectly competitive equilibrium determined at the initial point in time that remains the same for all future periods, in the model developed in this work it will be considered a series of equilibrium states of economic system. General economic equilibrium models with the asset markets should provide the possibility of analyzing income transfers between time periods, and the process of asset prices formation. After defining the appropriate general economic equilibrium model which involves asset markets in the absence of arbitrage process, further analysis is focused on the examination of Pareto optimality of economic system.

With the aim of making our model more realistic, in our analysis of economic system we assume that certain level of uncertainty in the economic processes exists.

In our analysis, we look at two points in time, where at the initial point in time ($t=0$), consumers (agents) are not sure which of the states will be realized in the following period of time ($t=1$). The uncertainty that exists for consumers at time $t=1$ is viewed as the main event or state of nature.

In the sequential model, markets are open at, or just before, each point in time/space state as pair of time/ state. The market for time t and state s (market ts) including the point of time $t=0$ represents spot market. Consumer i buy commodity l at time/state ts using his income. We can present model of sequential structure of market through:

– system of spot commodity markets in a period $t=0$ and in each of the states $s \in S_1$ in a period $t=1$, and
– system of asset markets (mostly financial markets) where assets are instruments of income redistribution through the different states of nature of economic system

Depending on how dividends are paid, assets are divided into two classes - nominal assets, for which dividends are paid in units of account, and real assets, for which dividends are paid in the form of commodity bundle. Nominal assets are defined as a contract that promises delivery of a nominal value, which we present in the form of dividend vectors in each state $s \in S_1$ in period of time $t=1$. Suppose that collection $S=(1,\dots,J)$ is a set of assets. Based on defined set of assets and particular dividend vector, dividend value matrix could be defined in the following way:

$$V(s)=\left[v^j(s)\right]_{SxJ} \tag{1}$$

where dividend vectors are represented in the columns of the matrix which describes the asset structure. If we extend dividend value matrix, given by the expression (1), with the price vector at time $t=0$ we get extended dividend value matrix W:

$$W=\begin{bmatrix} -q \\ V(s) \end{bmatrix}. \tag{1a}$$

Matrix $V(s)$ generates subspace of income transfers $\langle V(s)\rangle$

[1] In the context of general equilibrium, sequential structure of the economic system was analyzed by Hicks (1939) and later by Stigum (1969), Grandmont (1974, 1977) and Drèze (2009).

$$\langle V(s)\rangle = \{ r \in R^{S} | r(s) = V(s)\theta,\ \theta \in R^{J} \} \quad (2)$$

Consumer portfolio θ is determined by net income vector r for all expected states of nature $s \in S_0$ as a linear combination of columns of extended dividend value matrix W:

$$r = W\theta \quad (3)$$

The linear subspace that crosses over columns of extended dividend value matrix represents subspace of income transfers

$$M = \langle W\rangle = \{ r \in R^{S+1} | r = W\theta \ \ for\ \theta \in R^{J} \} \ . \quad (4)$$

Since the rank of matrix $V(s)$ by rows is equal to the rank of matrix $V(s)$ by columns and that dividend value matrix $V(s)$ has S rows, the size of matrix $V(s)$ is less than or equal to the number of the expected states of nature $S_1 = \{1,2, \dots, S\}$[2]. In the next step, we define a complete and incomplete structure of asset markets in case of uncertainty.

Definition 1. We observe a set of expected states of nature $S_1 = \{1,2, \dots, S\}$. If dimension of subspace of net income transfers $\langle V(s)\rangle$ under uncertainty is equal to the expected number of states of nature S, *i.e.* $\dim\langle V(s)\rangle = S$, then such a structure is a complete asset market structure. Conversely, if dimension of the subspace of net income transfers $\langle V(s)\rangle$ under uncertainty is strictly less than the number of expected state of nature S, *i.e.* $\dim\langle V(s)\rangle < S$, then such a structure is an incomplete asset market structure.

With this definition we set up the basics of the asset market structure. If market structure is complete, consumer can achieve any net income vector for each state of nature of the economic system. Conversely, if asset market structure is incomplete, then at least one net income vector will be always unavailable to consumer. Since we introduced the assets in economic system in order to give agents (consumers) ability to transfer income through time using dividends earned by a specific portfolio, we also assume that there is an asset market at the initial point in time $t = 0$. However, we have not analyzed the problem of equilibrium prices formation on asset markets. The necessary condition for the existence of equilibrium of asset markets, and indirectly of commodity markets, is absence of arbitrage process. In other words, we have to show that absence of arbitrage implies the existence of the discount factors at time $t = 0$ such that the asset prices at time $t = 0$ equal to their discounted values, both for complete and incomplete asset market structure.

Suppose that the asset structure is given by dividend value matrix $V(s)$ (expression (1)) with a set of assets $S = (1, \dots, J)$, and let the vector $q \in R^{J}$ be the vector of asset prices. Now, we can define the conditions for the absence of arbitrage process.

Definition 2. Let a pair $(V(s), q)$ represents the asset structure and asset prices at time $t = 0$. Then pair $(V(s), q)$ represents **no** - arbitrage process if the intersection between subspace M and space R_{+}^{S+1} is a zero vector, *i.e.* $M \cap R_{+}^{S+1} = \{0\}$. In other words, pair $(V(s), q)$ represents no - arbitrage process if and only if for some portfolio $\theta \in R^{J}$

[2] Mathematically, the rank of matrix $V(s)$ is less than or equal to the number of expected state of nature S, *i.e.* $rankV(s) \leq S$.

$$1.\ V(s)\cdot\theta > 0 \Rightarrow q\cdot\theta > 0, \quad 2.\ V(s)\cdot\theta = 0 \Rightarrow q\cdot\theta = 0 \qquad (5)$$

Suppose that there is a dividend value matrix $V(s)$ and a vector of asset prices q so that a pair of $(V(s),q)$ is no-arbitrage process. Furthermore, if we choose any discount factor vector $\beta \in R_{++}^{s+1}$ (where at time $t=0$, $\beta(0)=1$), then for each asset $j \in S$ we have that the price of the asset j is given with the following expression:

$$q_j = -\sum_{s \in S_1} \beta(s) v^j(s) \ . \qquad (6)$$

With next theorem the above – mentioned statements are presented.

Theorem 3. (Fischer (1972), Ross (1976), The Existence of the Discount Factor (Fundamental Theorem of Asset Prices)) For each state of nature of economic system $s \in S_1$ there is a dividend value matrix $V(s)$, and at time $t=0$ there is a vector of asset prices q, $(q_1, q_2, \ldots, q_J)$, and let extended dividend value matrix W has a shape $W = \begin{bmatrix} -q \\ V \end{bmatrix}$. Then a pair $(V(s),q)$ represents no - arbitrage process if and only if for every state of nature of economic system $s \in S_0$ there is a discount factor vector $\beta \in R_{++}^{s+1}$ $(\beta(0)=1)$ such that $\beta W = 0$, *i.e.* $q = \beta \cdot V(s)$.

If you continue to analyze the first row of extended dividend value matrix W at time $t=0$, we see that for each asset $j \in S$ asset price q_j represents the discounted value of dividend at time $t=1$. Therefore, if the pair $(V(s),q)$ represents no - arbitrage process, then rank of the extended dividend value matrix W equals to rank of the dividend value matrix $V(s)$, i.e. $rank\, W = rank\, V(s)$. In addition, if asset market structure is complete, discount factor vector is uniquely determined.

Consequence 4. Let a pair $(V(s),q)$ represents no - arbitrage process. If there is a complete asset market structure such that $rankW = S$, then:

(i) there is a unique discount factor vector $\beta \in R_{++}^{s+1}$ $(\beta(0)=1)$ such that $\beta W = 0$

(ii) net income linear subspace is $M = \{r \in R^{s+1} | \beta r = 0\}$.

Collection of discount factor vectors $\beta \in R_{++}^{s+1}$ is orthogonal to each column vector of the matrix W and represents orthogonal subspace of space $\langle W \rangle$. It follows that

$$\langle W \rangle^{\perp} = \{\beta \in R_{++}^{s+1} | \beta W = 0\} = \{\beta \in R_{++}^{s+1} | \beta r = 0, \ \forall r \in \langle W \rangle\} \ . \qquad (7)$$

The orthogonal subspace $\langle W \rangle^{\perp}$ in fact represents the space of present value vectors. In Consequence 4 we have established the existence of a single discount vector $\beta \in R_{++}^{s+1}$ in the case of complete asset market structure. It remains an open question the existence of a discount factor $\beta \in R_{++}^{s+1}$ if asset market structure is incomplete, in which no - arbitrage portfolio θ is given by Definition 2. Following theorem provides a description of subspace of income transfers in absence of arbitrage when asset market structure is incomplete.

Theorem 5. (Magill and Quinzii (2001)) Let a pair $(V(s),q)$ represents no - arbitrage process and W isextended dividend value matrix. If the rank of the extended dividend value matrix equals to the total number of means J, i.e. $rankW = J$ and $J \le S$, then there is $H = (S+1) - J$ linearly independent discount factor vectors $(\beta^1, \beta^2, \dots, \beta^H) \in R^{S+1}_{++}$, where at time $t = 0$, $\beta^h(0) = 1$, and for $h = 1,2, \dots, H$, $\beta^h W = 0$.

This theorem asserts the existence of the income transfers subspace in absence of arbitrage when asset market structure is incomplete. If the rank of the extended dividend value matrix $rankW = J$, then exist H linearly independent vectors that form a basis of vector subspace which is orthogonal to the subspace M. Based on specified orthogonal vector subspace $M^\perp = \langle W \rangle^\perp$, we define, in absence of arbitrage $(V(s), q)$ when asset market structure is incomplete, a subspace of income transfers M which has a shape $M = \langle W \rangle = \{ r \in R^{S+1} | \beta^1 r = \beta^2 r = \dots = \beta^H r = 0 \}$.

We can now introduce the basic assumptions concerning the consumption set, the vector of initial consumer wealth and consumer utility function.

Basic assumptions[3] :

(A1) $C = R^{L(S+1)}_{++}$ where $R^{L(S+1)}_{++}$ represents positive ortrant of commodity space;

(A2) $e \in R^{L(S+1)}_{++}$;

(A3) $u \in C^2(C, R)$, where $C^2(C, R)$ is a set of functions with domain C taking values from the set R

and have twice differentiable property;

(A4) for each $c \in R^m_{++}$, $Du(c) \in R^{L(S+1)}_{++}$ is Jacobian function of u;

(A5) for every $c \in R^m_{++}$, $\sum_{j=1}^{m} \sum_{k=1}^{m} h_j h_k \frac{\partial^2 u(c)}{\partial c_j \partial c_k} < 0$, and for every $h \in R^m$ and $h \neq 0$, then $\sum_{j=1}^{m} h_j \frac{\partial u(c)}{\partial c_j} = 0$,

where $D^2 u(c)$ represents the Hessian function of u in point c.

Since we introduced the basic assumptions, we can now formulate a mathematical concept of consumer that satisfies the established assumptions and of economic system with the asset structure in case that uncertainty exists.

Definition 6. Consumer is defined as ordered triplet (C, u, e), where $u : C \to R$ is utility function of consumer.

Definition 7. Economic system with the asset structure under uncertainty is determined with ordered triplet $(C^i, u^i, e^i)_{i \in N}$ where for every $i \in N$, ordered triplet (C^i, u^i, e^i) is consumer, and with the dividend value matrix specified by expression (1). It could be represented in the following form:

$$\Xi = \left\{ \left(C^i, u^i, e^i \right)_{i \in N}, V(s) \right\} \tag{8}$$

In further analysis, it is necessary to determine the portfolio selection that ensures consumer expenditure plan. Next theorem gives conditions that must be fulfilled by economic system Ξ if portfolio selection could be able to ensure consumption plan.

Theorem 8 (Magill and Quinzio (2002)). Let $\Xi = \left\{ \left(C^i, u^i, e^i \right)_{i \in N}, V(s) \right\}$ be economic system with the asset structure and let a pair $\left(\left(\bar{c}^i \right)_{i \in N}, p \right)$ represents the spot market equilibrium of economic system Ξ, where the subspace of income transfers is equal to the extended dividend value matrix,

[3] Detailed review of implications of basic assumptions is given in Balasko (1988).

i.e. $M = \begin{bmatrix} -q \\ V(s) \end{bmatrix} = \langle W(p,q) \rangle$. If for $q \in R^{S+1}$ and for $i \in N, s \in S_0$ holds $\bar{r}^i(s) = p(s)\left[\bar{c}^i(s) - e^i(s)\right]$, then there is selection of portfolio θ^{-i} for consumer $i \in N$ such that:

(i) $\sum_{i \in N} \bar{\theta}^i = 0$ (Sum of consumer portfolio selections in economic system is equal to zero), and for the consumer $i \in N$ it holds $\bar{r}^i = W(p,q)\bar{\theta}^i$, so that the selection of portfolio $\bar{\theta}^i$ is a solution of **consumer portfolio selection problem,** which has the following form:

$$\begin{aligned} &\max_{\theta^i} v^i\left(p, r^i(0), r^i(1), \dots, r^i(S)\right) \\ &r^i = W(p,q)\theta^i \end{aligned} ; \tag{9}$$

(ii) for consumer $i \in N$ with condition $\bar{r}^i = W(p,q)\bar{\theta}^i$, consumption plan $\bar{c}^i$ represents solution of **consumer portfolio selection problem on spot - markets,** which is given in the following form:

$$\begin{aligned} &\max_{c^i} u^i\left(c^i(0), c^i(1), \dots, c^i(S)\right) \\ &c^i \in C^i, \\ &\text{for } s \in S_0, \quad p(s)\left(c^i(s) - e^i(s)\right) \le \bar{r}^i(s) \end{aligned} . \tag{10}$$

The conditions of Theorem 8 imply that when the economic system is in equilibrium, where the set of net income vectors is generated by the assets, the ordered pair $\left(\bar{c}^i, \bar{\theta}^i\right)$ represents an equilibrium solution of consumer problem with the following form:

$$\begin{aligned} &\max_{(c^i,\theta^i)} u^i\left(c^i(0), c^i(1), \dots, c^i(S)\right) \\ &c^i \in C^i, \\ &\text{for } s \in S_0, \ p(s)\left[c^i(s) - e^i(s)\right] \le r^i(s) \\ &\text{and,} \qquad r^i = W(p,q)\theta^i \end{aligned} . \tag{11}$$

Theorem 8 claims that a collection of net income vectors $M = \langle W(p,q) \rangle$, as a subspace of income transfers, is generated by the assets in economic system.

2. THE EXISTENCE OF SOLUTION IN THE SEQUENTIAL MODEL OF ECONOMIC SYSTEM

Having defined the economic system with asset structure under uncertainty, our further analysis will be focused on the study of the existence of general economic equilibrium of incomplete market structure with the dividend payment in the nominal value. With Theorem 8 we decompose consumer problem in the real and financial flows of economic system. We will now explore the existence of equilibrium of the financial system and the existence of no - arbitrage equilibrium of economic system.

Consumer as the owner of the portfolio $\theta = (\theta_1, \theta_2, \dots, \theta_J) \in R^J$ earns an appropriate dividend given in the form of a dividend value matrix $V(s)$ for realized states $s = 1,2,\dots,S$ of economic system at time $t=1$. Vector of asset prices $q=(q_1,q_2,...,q_J)$ at time $t=0$, added as the first row of dividend matrix $V(s)$ forms an extended dividend value matrix W, based on which we can define budget set for the consumer i in the form of

$$B_F=\{c^i\in C \mid c^i-e^i=W\theta^i,\quad \theta^i\in R^J\}. \tag{12}$$

Consumer i chooses the pair of vectors - the consumption vector c^i and portfolio vector θ^i. Thus, a pair (c^i,θ^i) represents choice of consumer i, where portfolio vector θ^i is such that

$$c^i-e^i=W\theta^i. \tag{13}$$

Therefore, portfolio vector θ^i shows a method of financing the consumption vector c^i. Equilibrium of financial system is given by the following definition.

Definition 9. The equilibrium of the financial system $\Xi_F=\left\{\left(C^i,u^i,e^i\right)_{i\in N},V(s)\right\}$ represents a pair of two actions $\left(\bar{c}^i,\bar{\theta}^i\right)$ and vector of asset prices $\bar{q}$, such that the following relations are satisfied:

(i) for $i=1,2,...,I,\ \left(\bar{c}^i,\bar{\theta}^i\right)\in\arg\max\left\{u^i\left(c^i\right)\middle|\left(c^i,\theta^i\right)\in B_F\right\}$;

(ii) for each $i=1,2,....,I$, $V(s)\bar{\theta}^i=\bar{c}_1-e_1^i$;

(iii) $\sum_{i=1}^I\bar{\theta}^i=0$.

Thus, market equilibrium equations $\sum_{i=1}^I\bar{\theta}^i=0$ imply that the consumption allocation $\bar{c}^i$ is feasible if $\sum_{i=1}^I\left(\bar{c}^i-e^i\right)=0$. Now, when the equilibrium of the financial system is defined, and since there is a discount factor $\beta=(\beta(0),\beta(1),\dots,\beta(S))$ which is normalized such that $\beta(0)=1$, and there is no - arbitrage process $q=\beta V$, we can perform the following transformations of budget constraints, *i.e.* budget constraints of the budget set B_F at time $t=0$, are $p(0)\left(c^i(0)-e^i(0)\right)=-q\theta^i=-\beta V(s)\theta^i=-\sum_s\beta(s)V(s)\theta^i$; budget constraints of the budget set B_F at time $t=1$, are $\sum_s\beta(s)p(s)(c^i(s)-e^i(s))=\sum_s\beta(s)V(s)\theta^i$. Once transformations are introduced, we can define no-arbitrage equilibrium of economic system.

Definition 10. No-arbitrage equilibrium of economic system $\Xi_F=\{(C^i,u^i,e^i)_{i\in N},V\}$ represents a pair of vectors $\left(\bar{P},\bar{c}\right)$ that satisfies the following:

(i) for each i, the consumption vector $\bar{c}^i$ is a solution to the following problem:

$$\max_{c^i}u^i\left(c^i\right),$$

$$P\left(c_0^i-e_0^i\right)=0$$

$$P_1\left(c_1^i-e_1^i\right)\in\left\langle\beta_1V(s)\right\rangle,$$

(ii)

$$\sum_{i=1}^I c^i=\sum_{i=1}^I e^i.$$

Having defined the equilibrium of the financial system and the equilibrium of no - arbitrage economic system, the next theorem gives the relationship between these two equilibriums.

Theorem 11 (Magill and Quinzio (2002)). Arbitrage equilibrium of the economic system $\Xi = \left\{ \left(C^i, u^i, e^i \right)_{i \in N}, V(s) \right\}$ exists if and only if there is equilibrium of the financial system.

Theorem 11 claims that there is interdependence between equilibrium model of the financial system and the equilibrium model of no-arbitrage economic system. Our further analysis is focused on the application of direct access to the proof of existence of equilibrium solution in the model of the financial system. Furthermore, in the proof of existence of equilibrium in the model of the financial system we limit our analysis to the case of no - arbitrage market exchange processes. The importance of introducing restrictive assumptions about the absence of arbitrage in the process of asset prices formation is clearly evident in the following well - known theorem.

Theorem 12 (Magill and Quinzio (2002)). Suppose that $\Xi_F = \left\{ \left(C^i, u^i, e^i \right)_{i \in N}, V(s) \right\}$ is the financial system, and that utility function u^i satisfies the assumptions (A3) - (A5). Then, the following conditions are equivalent:

(i) for each consumer $i = 1,2, \ldots, I$, consumption vector c^i is solution of the problem of maximizing utility function u^i on the financial budget set B_F i.e. for each $i = 1,2, \ldots I$, $\arg\max \{ u^i(c^i) | c^i \in B_F \} \neq 0$;

(ii) vector of asset prices $q \in R^J$ represents no - arbitrage process

(Definition 2.);

(iii) there is discount factor vector $\beta \in R^{S+1}_{++}$ such that $q = \beta V$, i.e. $\beta W(q, V) = 0$;

(iv) financial budget set is compact set for each consumer $i = 1,2, \ldots, I$.

Theorem 12, as a fundamental theorem of asset pricing, claims that for consumer (agent) with a monotone utility function there is solution of the problem of maximum if and only if no - arbitrage condition is satisfied on the financial markets. Specifically, for each financial instrument j, discount factor vector $\bar{\beta}$ defines profit from purchasing one unit of asset j by price q_j at time $t = 0$. Moreover, at time $t = 1$, dividend V_s^j is paid if the state s ($s = 1,2, \ldots, S$) occurs. Consequently, the previous theorem states that the existence of arbitrage process is equivalent to the existence of a positive discount factor vector.

In the rest of the paper, we will limit our analysis on a set of asset prices formed in the absence of arbitrage. The set of no – arbitrage prices is defined in the following way:

$$Q = \left\{ q_F = (p(0), q) \in R^{J+1} \middle| p(0) > 0, \exists_{\beta \in R^S_{++}} : q = \beta V \right\}. \tag{14}$$

In order to prove the existence of an equilibrium of the financial system, we define a demand function for financial instruments:

$$f^i(q_F) = \arg\max \{ u^i(c^i) | c^i \in B_F \}. \tag{15}$$

Now, on the basis of established basic assumptions (A3) - (A5) of the utility function of consumers i and of Berge theorem of maximum (Border (1985)), and since the budgetary correspondence $B_F : Q \to R^{S+1}_{+}$ is half - continuous from above and below, we have that the demand function $f^i(q_F)$ is continuous on the set Q. Furthermore, as basic tool in the proof of existence of equilibrium, we define aggregate excess demand function $Z : Q \to R^{J+1}$ as:

$$Z(q_F)=(Z_0(q_F),Z_1(q_F))=\left(\sum_{i=1}^{I}(f_0^i(q_F)-e_0^i),\sum_{i=1}^{I}\theta^i(q_F)\right). \quad (16)$$

Based on established basic assumptions (A1) - (A5) for utility function, and on the assumption that rank of the dividend value matrix is equal to the number of assets, *i.e.* $\dim[V(s)]=J$, we can conclude that aggregate excess demand function $Z:Q\to R^{J+1}$ satisfies the following properties:

(i) aggregate excess demand function $Z(q_F)$ is continuous on the set Q ;

(ii) for each $\alpha>0$ and $q_F\in Q$, aggregate excess demand function is homogeneous of zero degree, *i.e.* $Z(\alpha q_F)=Z(q_F)$;

(iii) for each $q_F\in Q$, $\left(Z_0(q_F),V(s)Z_1(q_F)\right)\geq -e$, (restriction of set from below);

(iv) for each $q_F\in Q$, $q_F Z(q_F)=0$ (stronger version of Walras's Law).

Now, since we introduced the aggregate excess demand function, the proof of the existence of equilibrium of the financial system $\Xi_F=\left\{\left(C^i,u^i,e^i\right)_{i\in N},V(s)\right\}$ is implemented in the area of asset prices and consumer commodity prices $\bar{Q}$ at $t=0$, wherein

$$\bar{Q}=\left\{q_F=\left(p(0),q\right)\in R^{J+1}\,\middle|\,p(0)\geq 0,\exists_{\beta\in R_+^S}:q=\beta V(s)\right\}. \quad (17)$$

To prove the existence of equilibrium of financial system we use statements that are contained in the proof of existence of equilibrium solutions given by Talman and Thijssen (2004).

Theorem 13 (Talman and Thijssen (2004)). Suppose that $\Xi_F=\left\{\left(C^i,u^i,e^i\right)_{i\in N},V(s)\right\}$ is the financial system, and that basic assumptions (A1) - (A5), as well as the assumption that rank of the dividend value matrix is equal to the number of assets, i.e. $\dim[V(s)]=J$, are satisfied. Then, there is an equilibrium solution in the model of the financial system with a vector of no – arbitrage prices $\bar{q}_F\in Q$.

Proof. Vector of no – arbitrage prices $\bar{q}_F\in Q$ represents the equilibrium vector (equilibrium of the financial system) if and only if the aggregate excess demand function $\bar{q}_F$, as a function of no – arbitrage prices, is equal to zero, i.e. if $Z(\bar{q}_F)=0$. We define for each $\nu\in N$ set Q^ν such as non-empty, convex and compact set:

$$Q^\nu=\left\{\begin{array}{l} q_F\in\tilde{Q}\left|\dfrac{1}{\nu\left(\bar{\theta}_0+bV(s)\bar{\theta}_1\right)}\leq p(0)\leq\dfrac{\nu}{\bar{\theta}_0+bV(s)\bar{\theta}_1},\quad q=\beta V(s)\right. \\ \forall j,\quad \dfrac{1}{\nu\left(\bar{\theta}_0+bV(s)\bar{\theta}_1\right)}\leq\beta_j\leq\dfrac{\nu}{\bar{\theta}_0+bV(s)\bar{\theta}_1}\end{array}\right\}$$

wherein, for each normalized vector of no – arbitrage prices $q_F\in\tilde{Q}$, there is some $n\in N$ that for each $\nu\geq n$ vector of normalized no – arbitrage prices belongs to Q^ν , i.e. $q_F\in Q^\nu$. Since we start from the fact that the demand function $f^i(q_F)$ represents correspondence[4], we opt for Kakutani's

[4] Thus, aggregate excess demand function $Z(q_F)$ represents correspondence.

fixed-point theorem, which is the extended form of Brower's fixed point theorem. Given that the aggregate excess demand function $Z(q_F)$ is continuous on the set $\bar{Q}$, and therefore on each of the sets $(Q^\nu)_{\nu\in N}$, then on the basis of a fixed point theorem, we can conclude that there is a fixed point $\bar{q}_F \in Z(\bar{q}_F)$ in each of the sets $(Q^\nu)_{\nu\in N}$. Respectively, for each $\nu \in N$, exists $\bar{q}_F \in Q^\nu$ so that for each vector of normalized no – arbitrage prices $q_F \in Q^\nu$, $q_F Z\left(\bar{q}_F\right) \le \bar{q}_F Z\left(\bar{q}_F\right) = 0$. Thus, the aggregate excess demand function should fulfill Walras's Law, i.e. for each $q_F \in Q$, $q_F Z(q_F) = 0$. Sequence $(q_F^\nu)_{\nu\in N}$ of area $\bar{Q}$ is limited and there is a subsequence that converges toward the point $\tilde{q}_F$, which belongs to the border of set $\partial\bar{Q}$. In addition, the aggregate excess demand function tends to infinity, i.e. if $\bar{q}_F \in \partial\bar{Q}$ we have $Z_0\left(q_F^\nu\right) + bV(s)Z_1\left(q_F^\nu\right) \to \infty$, and there is some $n \in N$ so that, for each $\nu \ge n$ $Z_0\left(q_F^\nu\right) + bV(s)Z_1\left(q_F^\nu\right) > 0$. Then, since the normalized vector of no - arbitrage normalized prices $q_F = (p(0), q) \in Q^\nu$, is given by conditions $p(0) = \dfrac{1}{\bar{\theta}_0 + bV\bar{\theta}_1}$, $q = \dfrac{bV(s)}{\bar{\theta}_0 + bV(s)\bar{\theta}_1}$, we get that

$p(0)Z_0\left(q_F^n\right) + qZ_1\left(q_F^n\right) = \dfrac{Z_0\left(q_F^n\right) + bV(s)Z_1\left(q_F^n\right)}{\bar{\theta}_0 + bV(s)\bar{\theta}_1} > 0$, which is contradictory with the fact that $q_F^n \in Q^n$ represents a fixed point for which $q_F^n \in Z(q_F)$. Thus, a series $(q_F^\nu)_{\nu\in N}$ is limited, and converges to the point $\bar{q}_F$ which belongs to the interior of $\bar{Q}$, i.e. $\bar{q}_F \in \bar{Q}^0 \subset Q$[5]. In addition, since the mapping $Z(q_F)$ is given in the form of a linear transformation $\left(Z_0\left(q_F\right), V(s)Z_1\left(q_F\right)\right)$ (where the matrix $V(s)$ is J - dimensional square matrix), we have a linear problem, whereby $\bar{q}_F$ represents a solution to the linear programming problem, whose primary form is $\max\left\{q_F Z\left(\bar{q}_F\right)\right\}$, *s.t.* $q_F\bar{\theta} = 1$, and dual is $\min\{\lambda\}$, *s.t.* $\lambda\bar{\theta} = Z\left(\bar{q}_F\right)$. Based on Walras's Law $\bar{q}_F Z\left(\bar{q}_F\right) = 0$ and solutions of primary and dual form of linear programming problem, it could be obtained that $\lambda\bar{q}_F\bar{\theta} = \lambda$ and $\bar{q}_F Z\left(\bar{q}_F\right) = \lambda \iff \lambda = 0$. We can conclude that the aggregate excess demand function, for the equilibrium price vector as a certain fixed point, is equal to zero, i.e. $Z\left(\bar{q}_F\right) = 0$, which is supposed to be proved.

Thus, we have shown that there is equilibrium of the financial system with a normalized vector of no – arbitrage prices. Furthermore, since we perform decomposition of consumer problem into the real and financial flows (Theorem 8), and since we found that there is interdependence between equilibrium model of the financial system and no-arbitrage equilibrium model of the economic system (Theorem 11), and that there is an equilibrium solution of no - arbitrage equilibrium model, we deduce that there is an equilibrium in general economic equilibrium model with incomplete market structure.

2.1 Extended sequential model of economic system

Now we consider a more complex model of the economic system, where we introduce the assumption that there is central authority in the economic system through which money is put into the economic system. In the previous part, we observe market processes of economic system through two periods of time. Now, each of these time periods is divided into three periods in which we observe market processes of economic system. In the first period at time $t = 0$ each consumer i must sell a total initial wealth e^i to the central authority for which he receives the amount of

[5] The problem of determining a fixed point is known as the problem of complementarity (Eaves (1971).

money $m^i = pe^i$. In the second period at time $t=0$ consumers buy and sell assets and in the third period consumers buy commodities using money they received in previous period. At time $t=1$, for each state of nature of economic system, the same processes occur, expect in the second period, when consumers receive dividends from their assets.

Now we can define consumer problem in which money is involved:

$$\max_{(c^i,\theta^i)} u^i\left(c^i(0), c^i(1), \ldots, c^i(S)\right)$$
$$c^i \in C^i,$$
$$\text{for } s \in S_0,\ p(s)\left[c^i(s) - e^i(s)\right] \le r^i(s) \quad \text{and} \quad r^i = W(p,q)\theta^i,$$
$$\text{for } s \in S_0,\ m^i(s) = p(s)e^i(s)$$

(18)

If we compare the consumer problem defined by the expression (11) with the consumer problem given by the expression (18), we can notice that latter one contained additional system of constraints related to the amount of money generated by the sale of consumer initial wealth. The total amount of money in economic system for each state of nature is $M(s)$ and $\forall s \in S_0$ equals $M(s) = \sum_{i \in N} m^i(s)$. We can now precisely define the monetary equilibrium of the economic system with the asset structure under uncertainty.

Definition 14. Monetary equilibrium of economic system with the asset structure and money in case of uncertainty $\Xi = \left\{\left(C^i, u^i, e^i\right)_{i \in N}, V(s), M_m\right\}$, represents ordered n - tuple vector $\left(\left(\bar{c}^i, \bar{\theta}^i\right)_{i \in N}, p(s), q, M_m\right)$, where:

(i) ordered pair $\left(\bar{c}^i, \bar{\theta}^i\right)$ represents an equilibrium solution of consumer problem given by the expression (18);

(ii) commodity market equilibrium for all states of nature of economic system $s \in S_0$

$$\begin{aligned} \sum_{i \in N} \bar{c}^i(0) &= \sum_{i \in N} e^i(0) \\ \sum_{i \in N} \bar{c}^i(1) &= \sum_{i \in N} e^i(1) \\ &\vdots \\ \sum_{i \in N} \bar{c}^i(S) &= \sum_{i \in N} e^i(S) \end{aligned} \ ;$$

(iii) asset market equilibrium $\sum_{i \in N} \bar{\theta}^i = 0$;

(iv) $p(0)\sum_{i \in N} e^i(0) = M(0)$, $\forall s,\quad p(s)\sum_{i \in N} e^i(s) = M(s)$.

Further, based on theorem of existence of the discount factor (Theorem 3), there is a discount factor $\beta = (\beta(0), \beta(1), \ldots, \beta(S))$ normalized such that $\beta(0) = 1$. Since for no - arbitrage process $q = \beta V$, we can transform consumer problem given by expression (18) to no - arbitrage equilibrium model for consumer i:

$$\begin{aligned} &\max_{c^i} u^i\left(c^i\right) \\ &P\left(c_0^i - e_0^i\right) = 0 \\ &P_1\left(c_1^i - e_1^i\right) \in \left\langle \beta_1 V(s) \right\rangle \end{aligned} \qquad (19)$$

$$Pe^i = \beta m^i$$

After we define no - arbitrage equilibrium model for consumer i, we are going to define no - arbitrage equilibrium of economic system.

Definition 15. No - arbitrage equilibrium of economic system $\Xi_F = \left\{ \left(C^i, u^i, e^i \right)_{i \in N}, V(s), M_m \right\}$ represents a pair of vectors $\left(\bar{P}, \bar{c} \right)$ which satisfies:

(i) for each i, consumption vector $\bar{c}^i$ is solution to the problem given by expression (19);

(ii) commodity market equilibrium for all states of nature of economic system $s \in S_0$, $\sum_{i=1}^{I} c^i = \sum_{i=1}^{I} e^i$;

(iii) $\forall s \in S_0, \quad P \sum_{i \in N} e^i(s) = \beta M(s).$

In Theorem 8 we performed a decomposition of consumer problem into the real and financial flows of the economic system. Based on this decomposition, we formed no - arbitrage equilibrium model of the economic system, and the equilibrium model of the financial system. Furthermore, consumer problem which includes money, using theorem of the existence of the discount factor, has been transformed into a no - arbitrage equilibrium model. Now, using statements about the relationship between no - arbitrage equilibrium model of economic system and model of the financial system (Theorem 11), as well as the theorem that there is an equilibrium solution of financial system model (Theorem 13), we can conclude that there is monetary equilibrium in the model of economic system with the asset structure and money $\Xi = \left\{ \left(C^i, u^i, e^i \right)_{i \in N}, V(s), M_m \right\}$, under uncertainty, which has the form of ordered n - tuple of vectors $\left(\left(\bar{c}^i, \bar{\theta}^i \right)_{i \in N}, p(s), q, M_m \right)$.

3. CONSTRAINED PARETO OPTIMALITY OF SEQUENTIAL MODEL OF ECONOMIC SYSTEM

The assets offer consumers the ability to transfer income through the different states of nature of economic system. However, in the case of incomplete asset market structure, *i.e.* when there is a reduced ability to transfer income through the different states of nature of the economic system, the question is whether the equilibrium allocation are Pareto optimal allocation. For example, Borch (1962) in his work *"Equilibrium and reinsurance market,"* claimed that in case of incomplete asset market structure the existence of Pareto optimal allocation can not be guaranteed. Diamond (1967), in his paper *"The role of a stock market and a general equilibrium model with technological uncertainty",* showed that in the case of incomplete market structure with one product in each state of economic space competitive equilibrium is constrained Pareto optimal allocation. Introduced assumption is restrictive and does not represent real processes of economic system. However, in order to facilitate evidence and understanding of a particular form of Pareto optimality, we are forced to introduce some restrictions that differ from the actual processes in the economic system.

Constrained Pareto optimality consists in seeking ways of redistribution (reallocation) of assets by adjustments of prices and consumption allocation on spot markets that would lead to Pareto improvement[6]. Geanakoplos and Polemarchakis (1986), in their work *„Existence, Regularity, and Constrained Suboptimality of Competitive Asset Allocations when the Market is Incomplete"*, showed that the equilibrium of economic system with incomplete market structure is constrained

[6] Further elaboration of constrained Pareto optimality was given in the works of Geanakoplos and Polemarchakis (1986), Geankoplos, Magill, Quinzio and Dreze (1990), Herings and Polemarchakis (2003).

suboptimal. In doing so, they presented that certain Pareto improvements can be achieved by an appropriate redistribution of initial consumers wealth, and then introduced some restrictions on asset markets. So, instead of asking, "Is some equilibrium of economic system optimal?" it could be questioned "whether a particular consumption allocation is optimal when we imposed restrictions due to market imperfections?". Geanakoplos and Polemarchakis (1986), and later, Geankoplos, Magill, Quinzio and Dreze (1990), see economic policy measures of government (as centralized body) as a way for achieving Pareto improvements. However, the question is whether the government in the role of central planning authority with the measures of their economic policy can anticipate all the resulting adjustments on the spot markets as well as all individual utility effects.

Our further analysis is focused on simplified version of the general economic equilibrium with incomplete market structure that is presented by Magill and Quinzio in their first version of the work in 1996 (Magill and Quinzio (2002)). The aim is to investigate the possibility of existence of constrained Pareto optimality in imperfect market conditions. We start from the assumption that there are two period of time - $t=0$ (present) and $t=1$ (future), finitely large number of possible states of nature of economic system, finitely number of assets, where the structure of the dividend value matrix $V(s)$ is fixed, *i.e.* is exogenously given, and that there is only one consumption good in each state of nature in the spot markets of economic system. Also, we assume that the distribution of goods to consumers at time $t=1$ is realized on the basis of the selected portfolio. Financial instruments represent a fundamental mechanism for redistribution of risk at time $t=1$. The set of feasible consumption allocations with constraints is given by the next definition.

Definition 16. The consumption allocation $c=\left(c^1,c^2,\ldots,c^I\right)$ for economic system $\Xi=\left\{\left(C^i,u^i,e^i\right)_{i\in N},V(s)\right\}$ is V - feasible allocation if:

(i) consumption allocation $c=\left(c^1,c^2,\ldots,c^I\right)$ belongs to the set of feasible allocations $F=\left\{c\in C\left|\sum_{i=1}^{I}\left(c^i-e^i\right)\le 0\right.\right\}$, and

(ii) at $t=1$ for each consumer $i=1,2,\ldots,I$ is $c_1^i-e_1^i\in\langle V\rangle$, where F_V is set of V - feasible allocations:

$$F_V=\left\{c\in F\left|c_1^i-e_1^i\in\langle V(s)\rangle,\ \ i=1,2,\ldots,I\right.\right\}. \tag{20}$$

Constrained Pareto optimality can be now defined.

Definition 17. Consumption allocation $\bar{c}=\left(\bar{c}^1,\bar{c}^2,\ldots,\bar{c}^I\right)$ is constrained Pareto optimal allocation if it satisfies the following conditions:

(i) consumption allocation $\bar{c}=\left(\bar{c}^1,\bar{c}^2,\ldots,\bar{c}^I\right)$ belongs to set of V - feasible allocations F_V, *i.e.* $\bar{c}\in F_V$, and

(ii) **there is no** consumption allocation $c=\left(c^1,c^2,\ldots,c^I\right)$ which belongs to set of V - feasible allocations F_V ($c\in F_V$) such that for each $i=1,2,\ldots,I$,

$$u^i\left(c^i\right) \geq u^i\left(\bar{c}^i\right) \text{ and for some } i,\ u^i\left(c^i\right) > u^i\left(\bar{c}^i\right).$$

With imposed restrictions in the model and defined constrained Pareto optimality, we will now show that the model of the financial system with incomplete market structure is constrained Pareto efficient.

Theorem 18 (Grossman (1977)). Let a pair of actions $\left(\bar{c}^i, \bar{\theta}\right)$ and a vector of asset prices $\bar{q}$ represent the equilibrium of the financial system $\Xi_F = \left\{\left(C^i, u^i, e^i\right)_{i \in N}, V(s)\right\}$ (Definition 9). Then, consumption allocation $\bar{c} = \left(\bar{c}^1, \bar{c}^2, \ldots, \bar{c}^I\right)$ is constrained Pareto optimal allocation in space of dividend value matrix $V(s)$.

Proof. Suppose that there is consumption allocation $c = \left(c^1, c^2, \ldots, c^I\right) \in F_V$ such that for each $i = 1,2,\ldots,I$, $u^i\left(c^i\right) \geq u^i\left(\bar{c}^i\right)$, and for some i, $u^i\left(c^i\right) > u^i\left(\bar{c}^i\right)$, i.e. let assume that the consumption allocation $\bar{c} = \left(\bar{c}^1, \bar{c}^2, \ldots, \bar{c}^I\right)$ is not constrained Pareto optimal allocation. Since the consumption allocation $c = \left(c^1, c^2, \ldots, c^I\right) \in F_V$ then for every consumer $i = 1,2,\ldots,I$ there is portfolio $\theta^i \in R^J$, such that the consumption at $t = 1$ is $c_1^i = e_1^i + V(s)\theta^i$, and that the aggregate consumption at $t = 1$, $\sum_{i=1}^{I}\left(c_1^i - e_1^i\right) \leq 0$, which further implies that $V(s)\sum_{i=1}^{I}\theta^i \leq 0$. On the basis of a defined set of no - arbitrage prices Q (Expression (14)), and given conditions of equilibrium, we have that the vector of no - arbitrage price $\bar{q}$ is such that $\bar{q}\sum_{i=1}^{I}\theta^i \leq 0$. Since we assume that for some consumers i with consumption vector c^i, utility function is such that $u^i\left(c^i\right) > u^i\left(\bar{c}^i\right)$, consumption at $t = 0$ is $c_0^i > e_0^i - \bar{q}\,\theta^i$, which means that the consumption vector c^i does not belong to the budget set B_F (Expression (12)), i.e. $c^i \notin B_F$. Furthermore, based on assumption about the monotonicity of the utility function (assumption (A4)), for each consumer $j = 1,2,\ldots,I$, and $j \neq i$, $c_0^j \geq e_0^j - \bar{q}\,\theta^j$ holds. Then, we find that at time of $t = 0$ aggregate consumption is strictly greater than the aggregate initial wealth, i.e. $\sum_{i=1}^{I} c_0^i > \sum_{i=1}^{I} e_0^i - \bar{q}\sum_{i=1}^{I}\theta^i \geq \sum_{i=1}^{I} e_0^i$, which is contradictory for feasible consumption allocation c ■

Thus, we have shown that the consumption allocation, that satisfies requirements given by the Definition 17, is constrained Pareto optimal allocation.

Based on the assumptions of differentiability of utility functions (A3), and on the proof that consumption allocation is constrained Pareto optimal allocation in space of dividend value matrix (Theorem 18), we can apply the first and the second fundamental theorem of welfare economics (Arrow and Debreu (1954)) in the case of constrained Pareto optimality.

Namely, after discounting all spot values using the discount factor vector $\beta=(\beta(0),\beta(1),\ldots,\beta(S))$, the consumption allocation $\bar{c}=\left(\bar{c}^1,\bar{c}^2,\ldots,\bar{c}^I\right)$ is constrained Pareto optimal allocation if and only if every gradient of utility function is proportional to the normalized vector $\bar{\beta}^i=\left(1,\bar{\beta}_1^i\right)\in R_{++}^{S+1}$, *i.e.* for each $i=1,2,\ldots,I$, $grad\,u^i\left(\bar{c}^i\right)=\alpha_0^i\,\bar{\beta}_1^i\left(\bar{c}^i\right)$, $\left(\alpha_0^i=\dfrac{\partial c_0^i}{\partial u^i\left(\bar{c}^i\right)}\right)$.

Vector $\bar{\beta}_1^i\left(\bar{c}^i\right)$ represents the vector of the present value of consumption for consumer i where $\bar{c}^i$ is equilibrium consumption allocation which is constrained Pareto's optimal allocation. In addition, the normalized vector $\bar{\beta}_1^i$ is equal to the vector of present value $\bar{\beta}_1^i\left(\bar{c}^i\right)$, i.e. $\bar{\beta}^i=\beta^i\left(\bar{c}^i\right)$, and for each $i=1,2,\ldots,I$, $\bar{\beta}_1^i V(s)=\bar{q}$, *i.e.* $\beta_1^1\left(\bar{c}^1\right)V(s)=\beta_1^2\left(\bar{c}^2\right)V(s)=\ldots=\beta_1^I\left(\bar{c}^I\right)V(s)=\bar{q}$.

Based on the analysis carried out for the financial system $\Xi_F=\left\{\left(C^i,u^i,e^i\right)_{i\in N},V(s)\right\}$ with a pair of actions $\left(\bar{c}^i,\bar{\theta}\right)$ and a vector of asset prices $\bar{q}$, which represent the equilibrium of the system (given the Definition 9), the results can be represented as follows.

Consequence 19. For each consumer $i=1,2,\ldots,I$, portfolio $\bar{\theta}^i$ which financed consumption vector $\bar{c}=\left(\bar{c}^1,\bar{c}^2,\ldots,\bar{c}^I\right)\in F_V$ represents an optimal portfolio if for consumer i , $\bar{q}=\beta_1^i\left(\bar{c}^i\right)V(s)$ and if at the same time for each consumer $i=1,2,\ldots,I$ there is an equilibrium vector of asset prices $\bar{q}$, i.e. the requirement $\beta_1^1\left(\bar{c}^1\right)V(s)=\beta_1^2\left(\bar{c}^2\right)V(s)=\ldots=\beta_1^I\left(\bar{c}^I\right)V(s)=\bar{q}$ is satisfied. If we look at consumption at time $t=0$, (c_0^i), the portfolio choice θ^i for every consumer $i=1,2,\ldots,I$ and the indirect utility function which has a form $v^i\left(c_0^i,\theta^i\right)=u^i\left(c_0^i,e_1^i+V\theta^i\right)$, and apply statement Allocation of income transfers and Pareto optimal allocation (Dierker (1974), Mas-Colell (1985)) in case of constrained Pareto optimality, we can give the following consequences.

Consequence 20. The equilibrium consumption allocation $\left(\bar{c}^i\right)_{i\in N}$ is constrained Pareto optimal allocation with the equilibrium vector of asset prices $\bar{q}$

i.e. $\bar{q} = \beta_1^1 \left(\bar{c}^1\right) V(s) = \beta_1^2 \left(\bar{c}^2\right) V(s) = \ldots = \beta_1^I \left(\bar{c}^I\right) V(s)$, and multipliers $\lambda_0^i \in R$ are such that for each $i \in N$, gradients of indirect utility function, as a gradients of normalized portfolio, are proportional, i.e. for each *i*, $grad v^i(c_0^i, \theta^i) = \lambda_0^i q^i$ wherein

$$\lambda_0^i = \frac{\partial v^i\left(c_0^i, \theta^i\right)}{\partial c_0^i} \text{ and } grad\, v^i\left(c_0^i, \theta^i\right) = \left(\frac{\partial v^i\left(c_0^i, \theta^i\right)}{\partial \theta_1^i}, \ldots, \frac{\partial v^i\left(c_0^i, \theta^i\right)}{\partial \theta_J^i}\right).$$

Consequence 21. If the consumption allocation $\left(\bar{c}^i\right)_{i \in N}$ is constrained Pareto optimal allocation with its portfolio selection $\left(\bar{\theta}^i\right)_{i \in N}$ then on the basis of the ratio of partial derivatives of the utility function u^i and indirect utility function v^i, wherein $\frac{1}{\lambda_0^i} \frac{\partial v^i\left(c_0^i, \theta^i\right)}{\partial \theta_j^i} = \frac{1}{\lambda_0^i} \sum_{s=1}^{S} \frac{\partial u^i\left(c^i\right)}{\partial c_s^i} V_s^j$, and if the condition that $\beta_1^1 \left(\bar{c}^1\right) V(s) = \beta_1^2 \left(\bar{c}^2\right) V(s) = \ldots = \beta_1^I \left(\bar{c}^I\right) V(s) = \bar{q}$ is fulfilled, it follows that the normalized portfolio gradients $\bar{q}^i = \frac{1}{\lambda_0^i} grad\, v^i\left(\bar{c}_0^i, \bar{\theta}^i\right)$ in point $\left(\bar{c}_0^i, \bar{\theta}^i\right)$ are equal, and there is a vector of asset prices $\bar{q} \in R^J$ such that $\bar{q}^1 = \bar{q}^2 = \ldots = \bar{q}^I = \bar{q}$.

Consequence 22. For each consumer i, normalized portfolio gradient $q_j^i = \frac{1}{\lambda_0^i} \frac{\partial v^i\left(c_0^i, \theta^i\right)}{\partial \theta_j^i} = \frac{\partial c_0^i}{\partial v^i\left(c_0^i, \theta^i\right)} \frac{\partial v^i\left(c_0^i, \theta^i\right)}{\partial \theta_j^i}$ represents the marginal rate of substitution between asset j and the consumption at $t = 0$ and, as such, normalized portfolio gradient $\bar{q}^i$ on asset markets should lead to the equalization with the vector of asset prices $\bar{q}$ in any case when vector of asset prices $\bar{q}$ is not at equilibrium.

CONCLUSION

The model defined in this paper represents a model of sequential market structure consisting of spot commodity markets and the asset market, where the assets as instruments provide consumers (agents) the ability to redistribute income throughout the different states of nature of economic system. With the assumption that asset market structure is exogenously specified, and given the fact that the existence of incomplete asset market structure is economic reality, research is focused on determining whether solutions exist in general economic equilibrium model with incomplete market structure. Based on decomposition of general economic equilibrium model with asset market, we disaggregated general economic equilibrium model with incomplete asset market structure and nominal dividend matrix into two interrelated sub-models: general economic equilibrium model of real flows (*No-Arbitrage Equilibrium)* and asset markets equilibrium model (most commonly model of financial markets). Based on decomposition of general economic equilibrium model with incomplete asset market structure, we have proved that there is an equilibrium solution to the general economic equilibrium model of real flows if and only if there is an equilibrium solution to the asset markets equilibrium model. Evidence of the existence of equilibrium solutions have been carried out under the assumption of the absence of arbitrage in asset markets. The importance of introducing restrictive assumptions about the absence of arbitrage in asset markets is evident from the Fundamental Theorem of Asset Pricing.

Given the assumption that the structure of assets is exogenously determined, the research in this paper was primarily concentrated on inquiry into Pareto optimality of the defined general economic equilibrium model with incomplete asset markets. The space of income transfers is defined by the number of linearly independent vectors of dividend value matrix. In addition, the vector of income transfers belongs to the area of dividend value matrix, while the gradient of indirect utility function of consumers, which is the vector of marginal utility of income, belongs to the orthogonal subspace of space dividend value matrix. In the general economic equilibrium model with incomplete asset markets gradients of indirect consumers utility function are not proportional, i.e. there is no compatibility between present value vectors (there is no singular normalized discount factor vector). Thus, the equilibrium consumption allocation is not Pareto optimal allocation. The existence of imperfect market conditions in the form of incomplete asset market reduces possibility of income transfers throughout different states of nature of observed economic system, which results that equilibrium allocations are not Pareto optimal allocations.

The absence of Pareto optimal allocations in defined models required the examination of the allocation of consumption, which is Pareto optimal with the additional restrictions imposed, i.e. research has been focused on finding a limited form of Pareto's optimum in conditions of incomplete asset market. The set of feasible allocations of consumption has been redefined into the set of feasible allocations of consumption in the space of dividend value matrix. Thus defined set of achievable allocations of consumption, relative to space of dividend value matrix, we have defined limited Pareto optimal allocation of consumption and further proven that equilibrium solution to defined model of equilibrium on asset markets represent a limited Pareto optimal allocation relative to space of dividend value matrix. Furthermore, we have proven that equilibrium allocation of consumption is limited Pareto optimal allocation with equilibrium assets price vector if and only if the gradients of indirect utility function of all consumers are proportional as a normalized portfolio, i.e. normalized portfolio gradients represent marginal rate of substitution between investments in assets and consumption.

Based on these results, it can be concluded that development of asset market and adequate redistribution of consumers' wealth is necessary for every economic system that has a goal to obtain Pareto optimality. Additional research in this field should focus on more detailed analysis of influence of various economic policy measures on redistribution of consumer wealth, as well as development of asset markets and development and implementation of new forms of assets into economic system. Adequate implementation of economic policy measures would reduce influence of imperfect market conditions, with obtaining the goal of Pareto optimality of the observed economic system.

REFERENCES

Arrow, K., Debreu, G. (1954), "Existence of an Equilibrium for a Competitive Economy", *Econometrica*, Vol. 22, pp. 265-292.

Balasko, Y. (1988), *Foundations of the Theory of General Equilibrium*, Academic Press, Boston.

Borch, K. (1962), "Equilibrium and reinsurance market", *Econometrica*, Vol. 30, pp. 424-444.

Diamond, P. (1967),"The role of a stock market and a general equilibrium model with technological uncertainty", *American Economic Review*, Vol. 57, pp. 759-776.

Dierker, E. (1974), *Topological Methods in Walrasian Economics*, Springer Verlag, Berlin.

Border, K.C. (1985), *Fixed Point Theorems with Applications to Economics and Game Theory,* Cambridge University Press, Cambridge (UK).

Drèze, J.H., Lachiri, O., Minelli, E. (2009), “Stock prices, anticipations and investment in General Equilibrium,” *CORE Discussion Paper* 2009/83.

Eaves, B.C. (1971), “On the basic theorem of complementarity”, *Mathematical Programming 1*, North-Holland Publishing Company, pp. 68-75.

Fischer, S. (1972), “Assets, Contingent Commodities, and the Slutsky Equation”, *Econometrica*, Vl. 40, pp. 371-386.

Grandmont, J.M. (1974), “*On the Short Run Equilibrium in a Monetary Economy”*, in *Allocation under Uncertainty, Equilibrium and Optimality*, JH Dreze ed., Macmillan, London.

Geanakoplos, J.D., Polemarchakis, H.M. (1986), “Existence, Regularity and Constrained Suboptimality of Competitive Allocations Asset when the Market is Incomplete " in *WP Heller, RM Starr, and DA Starrett (eds.), Uncertainty, Information and Communication*: essays in Honor of KJ Arrow, Vol. III, Cambridge University Press, Cambridge, pp. 65-96.

Geanakoplos, J., Magill, M., Quinzio, M., Dreze, J. (1990), “Generic inefficiency of Stock Market Equilibrium When Markets are Incomplete”, *Journal of Mathematical Economics*, Vol. 19, pp. 113-151.

Grossman, S. (1977), “A Characterization of Optimality of Equilibrium in Incomplete Markets”, *Journal of Economic Theory*, Vol. 15, pp. 1-15.

Herings, P.J.J., Polemarchakis, HM. (2003), “Pareto improving price regulation when the asset market is incomplete”, The Birgit Grodal Symposium, Copenhagen, *Discussion Papers* 03-20.

Hicks, J. (1939), *Value and Capital*, Clarendon Press, Oxford.

Mas-Colell, A. (1985), *The Theory of General Economic Equilibrium, A Differentiable Approach*, Cambridge University Press, Cambridge.

Magill M., Quinzio M. (2002), *Theory of Incomplete Markets*, Vol. 1, The MIT Press Cambridge, Massachusetts London, England.

Ross, S.A. (1976), “The Arbitrage Theory of Capital Assets Pricing”, *Journal of Economic Theory*, Vol. 3, pp. 343-362.

Stigum, B. (1969), “Competitive Equilibrium under Uncertainty”, *Quarterly Journal of Economics*, Vol. 83, pp. 533-561.

Talman, D., Thijssen, J. (2004), *Existence of Equilibrium and Price Adjustments in a Finance Economy with Incomplete Markets*, Department of Econometrics & Operations Research and Center, Tilburg University, Tilburg.